An Axiomatisation of Duplication Equivalence in the Petri Box Calculus

Martin Hesketh and Maciej Koutny
Department of Computing Science
University of Newcastle upon Tyne
Newcastle upon Tyne NE1 7RU, United Kingdom

Abstract

The Petri Box Calculus (PBC) consists of an algebra of box expressions, and a corresponding algebra of boxes (a class of labelled Petri nets). A compositional semantics provides a translation from box expressions to boxes. There are several alternative ways of defining an equivalence notion for boxes, the strongest one being net isomorphism. In this paper we consider slightly weaker notion of equivalence, called *duplication equivalence*, which still can be argued to capture a very close structural similarity of concurrent systems the boxes are supposed to represent. We transfer the notion of duplication equivalence to the domain of box expressions and investigate the relationship between duplication equivalent boxes and box expressions. The main result of this investigation is a sound and complete axiomatisation of duplication equivalence for a fragment of recursion-free PBC.

Keywords: Petri nets, process algebra, equivalence, axiomatisation, theory of concurrency.

1 Introduction

Petri nets [17] and process algebras [1, 8, 10, 15] are two widely used and intensively researched models for concurrency. Petri nets have a partial order, or *true concurrency* behaviour, allowing reasoning about causal relationships between events. In comparison, process algebras are generally based on interleaving behaviours. The Petri net model, which is graphical in nature, does not readily support the composition of nets. This makes it more difficult to produce modular designs for systems than in process algebra based framework.

The Petri Box Calculus (PBC) [2, 4, 5, 14] has been designed to combine the advantages of both Petri nets and process algebras. The PBC consists of an algebra of box expressions, and a corresponding algebra of boxes (a class of labelled Petri nets). A compositional semantics provides a translation from box expressions to boxes. Earlier approaches to giving a Petri net interpretation to a process algebra [7, 16] have been based on algebras with an existing semantics in a model other than Petri nets. Note that the design of the PBC does not preclude a semantics being given in purely algebraic terms [12, 13]. For example, [13] gives a partial order structural operational semantics for box expressions which is consistent with the corresponding partial order semantics of boxes. PBC allows the semantics of high level programming constructs to be simulated, verified and reasoned about at the level of Petri nets. In this respect, the PBC lies midway between Petri nets and high level programming languages such as `occam` [11] and $B(PN)^2$ [3]. The PBC can be considered an extension of CCS [15], providing a more general synchronisation scheme and support for iteration.

One of the main motivations behind the development of the PBC was to make it possible to support in a single framework different verification techniques developed independently for process algebras and Petri nets. An example of the latter is the compositional S-invariant analysis [6]. In this paper we consider one of the standard verification techniques developed for process algebras, namely axiomatisation of behavioural equivalence [15].

There are several alternative ways of defining an equivalence notion for boxes, the strongest one being net isomorphism. In this paper we consider slightly weaker notion of equivalence, called *duplication equivalence* [2], which still can be argued to capture a very close structural similarity of concurrent systems expressed by boxes. In other words, duplication equivalent boxes should be equivalent with respect to every reasonable notion of behavioural equivalence. Thus duplication equivalence plays a fundamental role in the PBC approach and deserves a due investigation. In this paper we transfer the notion of duplication equivalence to the domain of box expressions and investigate the relationship between it and the structure of a box expression. The main result of this investigation is a sound and complete axiomatisation of duplication equivalence for a fragment of recursion-free PBC.

The paper is organised as follows. In section 2 we define a class of Petri nets which are used throughout the rest of this paper, and introduce the notion of duplication equivalence. Section 3 discusses the relationship between the operation of synchronisation and duplication equivalence. The next section introduces the remaining operators of the PBC. Section 5 defines box expressions and boxes, a class of Petri nets used in the PBC. The following section transfers the results on duplication equivalence obtained for boxes to the domain of box expressions. Section 7 contains the proposed axiomatisation of duplication equivalence. It is followed by the proofs of soundness and completeness of that axiomatisation, presented respectively in sections 8 and 9. Finally, we briefly discuss some of the issues related to the proposed axiomatisation.

2 Labelled nets

In what follows, we assume two infinite disjoint sets of elements, called *places* and *transitions* — jointly referred to as *nodes* — and an infinite set of *actions*. Any set of nodes N can be labelled using a *label function* which must return either one of the three special symbols (which are not actions), e, i and x, if applied to a place, or an action if applied to a transition. A *weight function* for N is a mapping from the Cartesian product $N \times N$ to the set of non-negative integers such that it always returns zero for a pair of places or a pair of transitions. We adopt the standard Petri net way of representing places and transitions as respectively circles and rectangles. A weight function will be represented using directed weighted arcs. No arc between two nodes will be drawn if the function returns zero.

We define a *net* to be a tuple $\mathcal{N} = (S, T, W, \lambda)$ such that S and T are finite sets of respectively places and transitions, and W and λ are respectively a weight and label function for the set of nodes $S \cup T$.

A place labelled by e is an *entry* place, by i an *internal* place, and by x an *exit* place. By convention, ${}^{\circ}\mathcal{N}$ and $\mathcal{N}^{\circ}$ will denote, respectively, the entry and exit places of $\mathcal{N}$. We will later use the entry and exit places to compose nets. For every node n in $\mathcal{N}$, we use ${}^{\bullet}n$ to denote its *pre-set* which comprises all the nodes m such that $W(m, n) \geq 1$. The *post-set* $n^{\bullet}$ is defined in a similar way. This notation extends in the usual way to every set of nodes N, e.g. ${}^{\bullet}N$ is the union of the pre-sets of the nodes in N. A transition t is *simple* if $W(s,t) \leq 1$ and $W(t,s) \leq 1$, for every place s. The net is *T-restricted* if the pre-set and post-set of every transition are non-empty. An *isolated* place is one whose pre-set and post-set are empty.

A *labelled net* is a T-restricted net without isolated places. Figure 1 shows a labelled net $\mathcal{N}$ such that

$$\begin{array}{rcl} S & = & \{s_0, s_1, s_2, s_3, s_4\} \\ T & = & \{t_0, t_1, t_2, t_3\} \\ \lambda & = & \{(s_0, \mathsf{e}), (s_1, \mathsf{e}), (s_2, \mathsf{i}), (s_3, \mathsf{i}), (s_4, \mathsf{x}), (t_0, a), (t_1, b), (t_2, c), (t_3, d)\} \end{array}$$

and $W(m, n) = 1$ for $(m, n) \in \{(s_0, t_0), (s_0, t_1), (s_1, t_1), (s_1, t_2), (s_2, t_3), (s_3, t_3), (t_0, s_2), (t_1, s_2), (t_1, s_3), (t_2, s_3), (t_3, s_4)\}$; and $W(m, n) = 0$ otherwise. $\mathcal{N}$ is T-restricted, its entry places are s_0 and s_1, and the exit place is s_4. We will later see that $\mathcal{N}$ corresponds to the box expression $((a \| c) \,\square\, b); d$, where $\|$ denotes parallel, $\square$ choice, and the semicolon sequential composition.

We will often decorate the different components of a net $\mathcal{N}$ with the index ${}_{\mathcal{N}}$. The same convention will apply to other notations we subsequently introduce. We will use the notation $n_1 \ldots n_k \bowtie m_1 \ldots m_l$

to mean that the 'sum' of the weight functions of nodes $n_1, \ldots, n_k$ is the same as the 'sum' of the weight functions of nodes $m_1, \ldots, m_l$. That is, for every node n in $\mathcal{N}$,

$$\sum_{i=1}^{k} W(n_i, n) = \sum_{i=1}^{l} W(m_i, n) \quad \text{and} \quad \sum_{i=1}^{k} W(n, n_i) = \sum_{i=1}^{l} W(n, m_i).$$

We will also say that $n_1, \ldots, n_k$ have the same *connectivity* as $m_1, \ldots, m_l$. Note that for the net of figure 1 we have $t_1 \bowtie_{\mathcal{N}} t_0 t_2$. To simplify some of the definitions we will use δ to denote a 'dummy' simple transition which, if present, would satisfy ${}^\bullet\delta = {}^\circ\mathcal{N}$ and $\delta^\bullet = \mathcal{N}^\circ$. For example, $t \bowtie_{\mathcal{N}} \delta u$ should be interpreted as signifying that $W(s,t) = W(s,u) + 1$, for all $s \in {}^\circ\mathcal{N}$, and $W(s,t) = W(s,u)$ otherwise; and that $W(t,s) = W(u,s) + 1$, for all $s \in \mathcal{N}^\circ$, and $W(t,s) = W(u,s)$ otherwise. We will say that nodes $n_1, \ldots, n_k$ have *constant connectivity* with a set of nodes N if, for all $n, m \in N$,

$$\sum_{i=1}^{k} W(n_i, n) \;=\; \sum_{i=1}^{k} W(n_i, m) \quad \text{and} \quad \sum_{i=1}^{k} W(n, n_i) \;=\; \sum_{i=1}^{k} W(m, n_i)$$

and denote this by $(n_1 \ldots n_k, N) \in \mathsf{const}_{\mathcal{N}}$. For example, in figure 1, $(t_0 t_2, \{s_0, s_1\}) \in \mathsf{const}_{\mathcal{N}}$. Directly from the definition of $\bowtie$ we obtain that

Proposition 2.1 If $n_1 \bowtie m_1, \ldots, n_k \bowtie m_k$ then $n_1 \ldots n_k \bowtie m_1 \ldots m_k$. Conversely, if $n_1 \ldots n_k \bowtie m_1 \ldots m_k$ and $n_1 \ldots n_{k-1} \bowtie m_1 \ldots m_{k-1}$ then $n_k \bowtie m_k$. $\square$

A *duplicate* of a node n of a net $\mathcal{N}$ is a node m which has the same label and connectivity as n, i.e. $\lambda(n) = \lambda(m)$ and $n \bowtie m$. We denote this by $n \simeq m$. Clearly, $\simeq$ is an equivalence relation; its equivalence class containing node n will be denoted by $[n]_\simeq$.

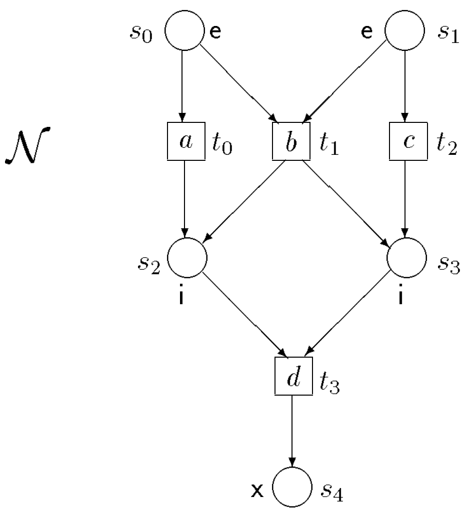


Figure 1: A labelled net.

2.1 Net isomorphism and net union

An *isomorphism* for two nets, $\mathcal{N}$ and $\mathcal{M}$, is a bijective mapping h from the nodes of $\mathcal{N}$ to the nodes of $\mathcal{M}$ such that for all nodes n, m in $\mathcal{N}$,

$$\lambda_{\mathcal{N}}(n) \;=\; \lambda_{\mathcal{M}}(h(n)) \quad \text{and} \quad W_{\mathcal{N}}(n,m) \;=\; W_{\mathcal{M}}(h(n), h(m)).$$

In other words, h is a graph isomorphism for $\mathcal{N}$ and $\mathcal{M}$ preserving node labelling. Note that since $\mathsf{e}, \mathsf{i}, \mathsf{x}$ are not actions, it is always the case that places are mapped onto places, and transitions are mapped onto transitions.

Net union is a partial operation defined only for pairs of *unionable* nets which means that their transition sets are disjoint and their label functions coincide on the common places. The *union* $\mathcal{N} \cup \mathcal{M}$ of two unionable nets, $\mathcal{N}$ and $\mathcal{M}$, is defined as a net with the node set being the union of the nodes of $\mathcal{N}$ and $\mathcal{M}$, and the weight and label functions being inherited from $\mathcal{N}$ and $\mathcal{M}$ (if the value for a weight of the new net cannot be found in the original nets, it is set to zero). Figure 2 shows two unionable nets, $\mathcal{N}_1$ and $\mathcal{N}_2$, and their union $\mathcal{N}_1 \cup \mathcal{N}_1$.

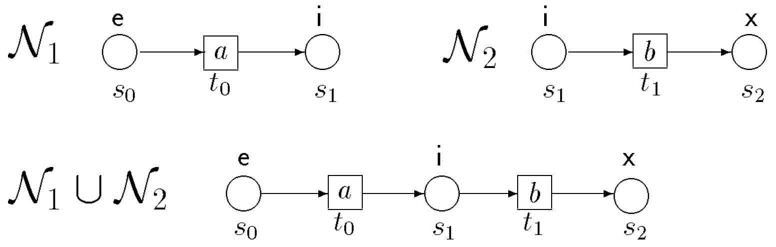


Figure 2: Net union.

Net union will usually be applied when the common places can be partitioned into $\otimes$-sets created by the operation of place multiplication (itself denoted by $\otimes$, see section 4). Let $\mathcal{N}$ and $\mathcal{M}$ be unionable nets. A non-empty set of places $P \subseteq S_\mathcal{N} \cap S_\mathcal{M}$ is a *$\otimes$-set* if for all $s, r \in P$ there is $p \in P$ such that $s \simeq_\mathcal{N} p$ and $r \simeq_\mathcal{M} p$.

Proposition 2.2 Let P be a $\otimes$-set for two unionable nets $\mathcal{N}$ and $\mathcal{M}$. If $t_1, \ldots, t_m$ are transitions in $\mathcal{N}$ and $v_1, \ldots, v_k, \ldots, v_{k+l}$ are transitions in $\mathcal{M}$ such that $m, l \geq 1$, $k \geq 0$ and

$$t_1 \ldots t_m v_1 \ldots v_k \bowtie_{\mathcal{N} \cup \mathcal{M}} v_{k+1} \ldots v_{k+l}$$

then ${}^\bullet t_1 \cup t_1{}^\bullet \cup \ldots \cup {}^\bullet t_m \cup t_m{}^\bullet \subseteq S_\mathcal{N} \cap S_\mathcal{M}$ and $(t_1 \ldots t_m, P) \in \mathsf{const}_\mathcal{N}$.

Proof. The first part follows directly from the definitions of $\bowtie$ and net union. To show $(t_1 \ldots t_m, P) \in \mathsf{const}_\mathcal{N}$, assume $s, r \in P$. Then, by P being a $\otimes$-set, there is $p \in P$ such that $s \simeq_\mathcal{N} p$ and $r \simeq_\mathcal{M} p$. Denote, for $x \in \{s, r, p\}$,

$$\kappa(x) \;=\; \sum_{i=k+1}^{k+l} W_\mathcal{M}(x, v_i) - \sum_{i=1}^{k} W_\mathcal{M}(x, v_i) \quad \text{and} \quad \mu(x) \;=\; \sum_{i=1}^{m} W_\mathcal{N}(x, t_i).$$

From $t_1 \ldots t_m v_1 \ldots v_k \bowtie_{\mathcal{N} \cup \mathcal{M}} v_{k+1} \ldots v_{k+l}$ it follows that $\mu(x) = \kappa(x)$ for $x \in \{s, r, p\}$. Moreover, from $s \simeq_\mathcal{N} p$ and $r \simeq_\mathcal{M} p$ we respectively obtain $\mu(s) = \mu(p)$ and $\kappa(r) = \kappa(p)$. Hence

$$\mu(s) = \mu(p) = \kappa(p) = \kappa(r) = \mu(r).$$

This and symmetry of the argument yield $(t_1 \ldots t_m, P) \in \mathsf{const}_\mathcal{N}$. $\square$

2.2 Duplication equivalence

Let $\mathcal{N}$ be a labelled net. We define the *duplication quotient* of $\mathcal{N}$ to be a labelled net

$$[\mathcal{N}]_\simeq \;=\; (\; \{\, [s]_\simeq \mid s \in S \} \;,\; \{\, [t]_\simeq \mid t \in T \} \;,\; W' \;,\; \lambda' \;)$$

where for all nodes n and m in $\mathcal{N}$, $\lambda(n) = \lambda'([n]_{\simeq})$ and $W(n,m) = W'([n]_{\simeq}, [m]_{\simeq})$. Figure 3 shows a labelled net $\mathcal{N}$ and its duplication quotient $[\mathcal{N}]_{\simeq}$. We now can introduce a notion central to this paper. Labelled nets $\mathcal{N}$ and $\mathcal{M}$ are *duplication equivalent* if their duplication quotients are isomorphic nets. We denote this by $\mathcal{N} \simeq \mathcal{M}$ or $\mathcal{N} \simeq_h \mathcal{M}$, where h is an isomorphism for $[\mathcal{N}]_{\simeq}$ and $[\mathcal{M}]_{\simeq}$. In the latter case we will also denote $n \simeq_h m$ if n and m are two nodes of respectively $\mathcal{N}$ and $\mathcal{M}$ such that $h([n]_{\simeq}) = [m]_{\simeq}$. As it was shown in [2], $\simeq$ is an equivalence relation. This can be slightly strengthened thus. For all labelled nets $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{P}$,

$$\mathcal{N} \simeq_h \mathcal{M} \text{ and } \mathcal{M} \simeq_g \mathcal{P} \text{ implies } \mathcal{N} \simeq_{h \circ g} \mathcal{P}. \tag{1}$$

Duplication equivalence has been introduced in [2] (under the name of renaming equivalence) as a structural equivalence relation on Petri nets used in PBC. It preserves the interleaving and non-interleaving (partial order) semantics of nets it identifies (provided we mark them initially by putting exactly one token in each of the entry places and no token on the other places[1]), and is a congruence with respect to all the net operators considered in [2] (hence also with respect to the net operators used in this paper).

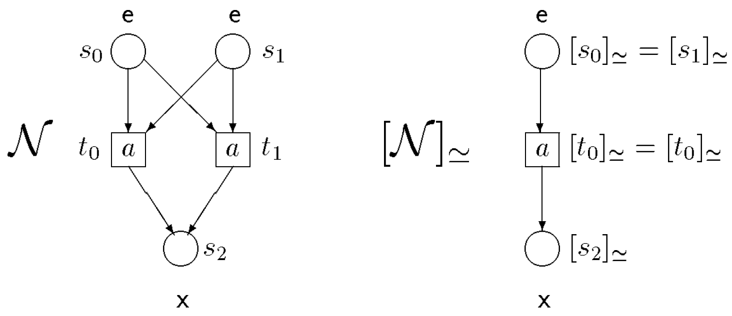


Figure 3: Labelled net and its duplication quotient.

The next result establishes some basic relationships between being a duplicate, having the same connectivity, and being duplication equivalent nets.

Proposition 2.3 Let $\mathcal{N} \simeq_h \mathcal{M}$ be duplication equivalent labelled nets.

1. If $n_1 \simeq_h m_1, \ldots, n_{k+l} \simeq_h m_{k+l}$ $(k, l \geq 1)$ and $n_1 \ldots n_k \bowtie_{\mathcal{N}} n_{k+1} \ldots n_{k+l}$ then $m_1 \ldots m_k \bowtie_{\mathcal{M}} m_{k+1} \ldots m_{k+l}$.
2. If $n \bowtie_{\mathcal{N}} n' \simeq_h m' \bowtie_{\mathcal{M}} m$ and $\lambda_{\mathcal{N}}(n) = \lambda_{\mathcal{M}}(m)$ then $n \simeq_h m$.
3. If $n \simeq_h m \bowtie_{\mathcal{M}} m' \simeq_{h^{-1}} n'$ then $n \bowtie_{\mathcal{N}} n'$.

Proof. (1) Let m be a node in $\mathcal{M}$. Then there is a node n in $\mathcal{N}$ such that $n \simeq_h m$. For every $i \leq k+l$, by $n_i \simeq_h m_i$, $W_{\mathcal{N}}(n, n_i) = W_{\mathcal{M}}(m, m_i)$. Hence

$$\sum_{i=1}^{k} W_{\mathcal{M}}(m, m_i) \;=\; \sum_{i=1}^{k} W_{\mathcal{N}}(n, n_i) \;=\; \sum_{i=k+1}^{k+l} W_{\mathcal{N}}(n, n_i) \;=\; \sum_{i=k+1}^{k+l} W_{\mathcal{M}}(m, m_i).$$

This and symmetry of the argument yield $m_1 \ldots m_k \bowtie_{\mathcal{M}} m_{k+1} \ldots m_{k+l}$.

[1] Such markings are the standard initial markings of boxes; we do not represent them explicitly here since we are concerned purely with the structure of the underlying nets.

(2) There is q such that $n \simeq_h q$. It suffices to show that $m \bowtie_{\mathcal{M}} q$. Take any node r in $\mathcal{M}$. We only prove that $W_{\mathcal{M}}(m,r) = W_{\mathcal{M}}(q,r)$. There is p in $\mathcal{N}$ such that $p \simeq_h r$. We have $W_{\mathcal{N}}(n,p) = W_{\mathcal{M}}(q,r)$ and $W_{\mathcal{N}}(n',p) = W_{\mathcal{M}}(m',r)$. Moreover, by $n \bowtie_{\mathcal{N}} n'$ and $m \bowtie_{\mathcal{M}} m'$, we have $W_{\mathcal{N}}(n,p) = W_{\mathcal{N}}(n',p)$ and $W_{\mathcal{M}}(m,r) = W_{\mathcal{M}}(m',r)$. Hence $W_{\mathcal{M}}(m,r) = W_{\mathcal{M}}(m',r) = W_{\mathcal{N}}(n',p) = W_{\mathcal{N}}(n,p) = W_{\mathcal{M}}(q,r)$.
(3) Suppose that p is a node in $\mathcal{N}$. Then there is r such that $p \simeq_h r$. By $m \bowtie_{\mathcal{M}} m'$, $W_{\mathcal{M}}(m,r) = W_{\mathcal{M}}(m',r)$. Moreover, by $n \simeq_h m$ and $n' \simeq_h m'$, we have $W_{\mathcal{N}}(n,p) = W_{\mathcal{M}}(m,r)$ and $W_{\mathcal{N}}(n',p) = W_{\mathcal{M}}(m',r)$. Hence $W_{\mathcal{N}}(n,p) = W_{\mathcal{N}}(n',p)$. Similarly, $W_{\mathcal{N}}(p,n) = W_{\mathcal{N}}(p,n')$. Thus $n \bowtie_{\mathcal{N}} n'$. □

2.2.1 Place-sharing nets

We will often be dealing with labelled nets $\mathcal{N}$ and $\mathcal{M}$ which are *place-sharing*, by which we mean that their place sets and place labellings are exactly the same (no conditions are imposed on the transition sets of the two nets). In such case, for all transitions, t in $\mathcal{N}$ and u in $\mathcal{M}$, we denote $t \simeq_{\mathcal{NM}} u$ if $\lambda_{\mathcal{N}}(t) = \lambda_{\mathcal{M}}(u)$ and for every place s in $\mathcal{N}$ (and so in $\mathcal{M}$),

$$W_{\mathcal{N}}(t,s) \;=\; W_{\mathcal{M}}(u,s) \quad\text{and}\quad W_{\mathcal{N}}(s,t) \;=\; W_{\mathcal{M}}(s,u).$$

Intuitively, $t \simeq_{\mathcal{NM}} u$ means that t and u are 'remote' duplicates since they have the same connectivity if one looks at the places of the two nets. Moreover, an isomorphism h for the duplication quotients of $\mathcal{N}$ and $\mathcal{M}$ establishing duplication equivalence of $\mathcal{N}$ and $\mathcal{M}$ will be called *place–preserving* if $s \simeq_h s$, for every place s in the two nets. We will denote this by $\mathcal{N} \cong_h \mathcal{M}$ or $\mathcal{N} \cong \mathcal{M}$. We now formulate a fundamental result concerning duplication equivalence of place-sharing nets.

Proposition 2.4 Let $\mathcal{N}$ and $\mathcal{M}$ be place-sharing labelled nets.

1. If $\mathcal{N} \cong_h \mathcal{M}$ then for all transitions, t in $\mathcal{N}$ and u in $\mathcal{M}$, $t \simeq_h u$ if and only if $t \simeq_{\mathcal{NM}} u$.
2. If $T_{\mathcal{N}} = \{t \mid t \simeq_{\mathcal{NM}} u\}$ and $T_{\mathcal{M}} = \{u \mid t \simeq_{\mathcal{NM}} u\}$ then $\mathcal{N} \cong \mathcal{M}$.

Proof. Denote $S = S_{\mathcal{N}} = S_{\mathcal{M}}$.
(1) Suppose $t \simeq_h u$. Then $\lambda_{\mathcal{N}}(t) = \lambda_{\mathcal{M}}(u)$. Let s be any place in S. Since $s \simeq_h s$, we have $W_{\mathcal{N}}(t,s) = W_{\mathcal{M}}(u,s)$ and $W_{\mathcal{N}}(s,t) = W_{\mathcal{M}}(s,u)$. Hence $t \simeq_{\mathcal{NM}} u$.
Suppose $t \simeq_{\mathcal{NM}} u$. Then $\lambda_{\mathcal{N}}(t) = \lambda_{\mathcal{M}}(u)$. Let $t \simeq_h w$ and s be a place in S. We have $W_{\mathcal{M}}(u,s) = W_{\mathcal{N}}(t,s) = W_{\mathcal{M}}(w,s)$ and $W_{\mathcal{M}}(s,u) = W_{\mathcal{N}}(s,t) = W_{\mathcal{M}}(s,w)$. Hence $u \simeq_{\mathcal{M}} w$ and so $t \simeq_h u$.
(2) Define a mapping from the nodes of the duplication quotient of $\mathcal{N}$ to the nodes of the duplication quotient of $\mathcal{M}$ thus:

$$h \;=\; \{([s]^{\mathcal{N}}_{\simeq}, [s]^{\mathcal{M}}_{\simeq}) \mid s \in S\} \;\cup\; \{([t]^{\mathcal{N}}_{\simeq}, [u]^{\mathcal{M}}_{\simeq}) \mid t \simeq_{\mathcal{NM}} u\}.$$

One can see that $\mathcal{N} \cong_h \mathcal{M}$. To show that h is a bijection, it suffices to show that: (i) if $s, r \in S$ and $s \simeq_{\mathcal{N}} r$ then $s \simeq_{\mathcal{M}} r$; and (ii) if t, u, t', u' are transitions such that $u \simeq_{\mathcal{MN}} t \simeq_{\mathcal{N}} t' \simeq_{\mathcal{NM}} u'$ then $u \simeq_{\mathcal{M}} u'$. To show (i), take any $y \in T_{\mathcal{M}}$. Then, by the assumption made in 2.4(2), there is $z \in T_{\mathcal{N}}$ such that $z \simeq_{\mathcal{NM}} y$. We thus have

$$W_{\mathcal{M}}(y,s) = W_{\mathcal{N}}(z,s) = W_{\mathcal{N}}(z,r) = W_{\mathcal{M}}(y,r).$$

The proof that $W_{\mathcal{M}}(s,y) = W_{\mathcal{M}}(r,y)$ is similar. To show (ii), take any $s \in S$. Then

$$W_{\mathcal{M}}(s,u) = W_{\mathcal{N}}(s,t) = W_{\mathcal{N}}(s,t') = W_{\mathcal{M}}(s,u').$$

The proof that $W_{\mathcal{M}}(u,s) = W_{\mathcal{M}}(u',s)$ is similar.
What remains to be shown is that if $s \in S$ and $t \simeq_{\mathcal{NM}} u$ then the weight of the arc between $[s]^{\mathcal{N}}_{\simeq}$ and $[t]^{\mathcal{N}}_{\simeq}$ in $[\mathcal{N}]_{\simeq}$ is the same as the weight of the arc between $[s]^{\mathcal{M}}_{\simeq}$ and $[u]^{\mathcal{M}}_{\simeq}$ in $[\mathcal{M}]_{\simeq}$. This, however, follows directly from $W_{\mathcal{N}}(s,t) = W_{\mathcal{M}}(s,u)$. □

Note that from the first part of the proposition it follows that there can be at most one place-preserving isomorphism (between the respective duplication quotients) establishing duplication equivalence of two place-sharing nets.

3 Synchronisation

To define a synchronisation operator on nets, we impose a little structure on the set of actions. We assume that it consists of *communication* actions, $\mathcal{A}$, and a distinct *internal* action, $\imath$. There is a bijection $\widehat{\ }: \mathcal{A} \to \mathcal{A}$ such that $\widehat{a} \neq a$ and $\widehat{\widehat{a}} = a$, for every a in $\mathcal{A}$. The actions a and $\widehat{a}$ will be called *conjugates*. By a *synchronisation set* we will mean a set of communication actions A which contains the conjugates of all its actions, i.e. $\widehat{A} = A$. For every communication action a, we will denote by a the synchronisation set $\{a, \widehat{a}\}$. As in CCS, it is implicitly assumed that two transitions labelled with conjugate communication actions can be synchronised to yield a new transition labelled with the internal action.[2] Two transitions, t and u, whose labels are conjugates belonging to a synchronisation set A are called *A–synchronisable*, $(t, u) \in \mathsf{syn}_A$.

The *synchronisation* of a labelled net $\mathcal{N}$ by a synchronisation set A is a net $\mathcal{N}\,\mathsf{sy}\,A$ which is defined as $\mathcal{N}$ extended by a set of new transitions. Exactly one new transition, $t \odot u$, is added for every pair of A–synchronisable transitions of $\mathcal{N}$, t and u. The label of $t \odot u$ is $\imath$ and the weight function is extended so that $t \odot u \bowtie_{\mathcal{N}\,\mathsf{sy}\,A} tu$. We also assume that $t \odot u$ is the same as $u \odot t$. Figure 4 shows two consecutive applications of the synchronisation operator. Note that $\mathcal{N}\,\mathsf{sy}\,\mathsf{a}$ and $(\mathcal{N}\,\mathsf{sy}\,\mathsf{a})\,\mathsf{sy}\,\mathsf{a}$ are duplication equivalent, but not isomorphic. Thus synchronisation is not idempotent with respect to net isomorphism. However, it is idempotent with respect to duplication equivalence which was one of the reasons for introducing duplication equivalence in [2].

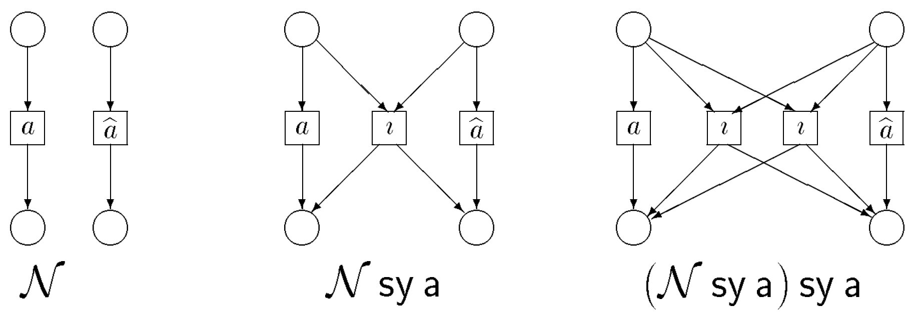


Figure 4: Synchronisation (place labels omitted).

Having the same connectivity is preserved through synchronisation.

Proposition 3.1 Let $\mathcal{N}$ be a labelled net with nodes $n_1, \ldots, n_k, m_1, \ldots, m_l$ and A be a synchronisation set. Then

$$n_1 \ldots n_k \bowtie_{\mathcal{N}} m_1 \ldots m_l \quad \text{if and only if} \quad n_1 \ldots n_k \bowtie_{\mathcal{N}\,\mathsf{sy}\,A} m_1 \ldots m_l.$$

Proof. That $(\Leftarrow)$ implication holds follows directly from the definition of synchronisation. Denote $\mathcal{M} = \mathcal{N}\,\mathsf{sy}\,A$. To show the reverse implication, suppose $t = u \odot w \in T_{\mathcal{M}} - T_{\mathcal{N}}$. By $n_1 \ldots n_k \bowtie_{\mathcal{N}} m_1, \ldots, m_l$,

$$\sum_{i=1}^{k} W_{\mathcal{N}}(n_i, u) = \sum_{i=1}^{l} W_{\mathcal{N}}(m_i, u) \quad \text{and} \quad \sum_{i=1}^{k} W_{\mathcal{N}}(n_i, w) = \sum_{i=1}^{l} W_{\mathcal{N}}(m_i, w).$$

[2] The synchronisation mechanism used in this paper is basically that of CCS; the original one used in PBC [2] is more general and can also express more complex, multi-way, synchronisation.

Hence, we obtain

$$\begin{aligned}
\sum_{i=1}^{k} W_{\mathcal{M}}(n_i,t) &= \sum_{i=1}^{k}(W_{\mathcal{M}}(n_i,u)+W_{\mathcal{M}}(n_i,w)) &&= \sum_{i=1}^{k} W_{\mathcal{N}}(n_i,u)+\sum_{i=1}^{k} W_{\mathcal{N}}(n_i,w)\\
&= \sum_{i=1}^{l} W_{\mathcal{N}}(m_i,u)+\sum_{i=1}^{l} W_{\mathcal{N}}(m_i,w) &&= \sum_{i=1}^{l}(W_{\mathcal{M}}(m_i,u)+W_{\mathcal{M}}(m_i,w))\\
&= \sum_{i=1}^{l} W_{\mathcal{M}}(m_i,t).
\end{aligned}$$

Similarly, one may show that $\sum_{i=1}^{k} W_{\mathcal{M}}(t,n_i)=\sum_{i=1}^{l} W_{\mathcal{M}}(t,m_i)$. □

Duplication equivalence is preserved through synchronisation. What is more, the isomorphism establishing the equivalence can also be preserved.

Proposition 3.2 Let $\mathcal{N} \simeq_h \mathcal{M}$ be duplication equivalent labelled nets and A be a synchronisation set. Then there is an isomorphism g such that $\mathcal{N}\,\mathsf{sy}\,A \simeq_g \mathcal{M}\,\mathsf{sy}\,A$ and, for all nodes n and m in respectively $\mathcal{N}$ and $\mathcal{M}$, $n \simeq_h m$ if and only if $n \simeq_g m$.

Proof. Denote $\mathcal{P}=\mathcal{N}\,\mathsf{sy}\,A$ and $\mathcal{R}=\mathcal{M}\,\mathsf{sy}\,A$. Define

$$g=\{\,([n]_{\simeq}^{\mathcal{P}},[m]_{\simeq}^{\mathcal{R}}) \mid n \simeq_h m\,\} \;\cup\; \{\,([t\odot u]_{\simeq}^{\mathcal{P}},[v\odot w]_{\simeq}^{\mathcal{R}}) \mid (t,u)\in \mathsf{syn}_A \wedge t \simeq_h v \wedge u \simeq_h w\,\}.$$

A straightforward yet rather lengthy argument can show that g is a required isomorphism. □

$\mathcal{N}$ and $\mathcal{N}\,\mathsf{sy}\,A$ are place-sharing nets. Hence, directly from proposition 2.4, we obtain a necessary and sufficient condition for the existence of a place-preserving isomorphism for the duplication quotients of $\mathcal{N}$ and $\mathcal{N}\,\mathsf{sy}\,A$.

Corollary 3.3 Let $\mathcal{N}$ be a labelled net and A be a synchronisation set. Then $\mathcal{N} \cong \mathcal{N}\,\mathsf{sy}\,A$ if and only if for every transition t in $\mathcal{N}\,\mathsf{sy}\,A$ there is u in $\mathcal{N}$ such that $t \simeq_{\mathcal{N}\,\mathsf{sy}\,A} u$.

Proof. The ($\Rightarrow$) implication follows from proposition 2.4(1), and the ($\Leftarrow$) implication from proposition 2.4(2). □

The next result can be interpreted as saying that being duplication equivalent to a synchronised net makes the synchronisation implicit.

Proposition 3.4 Let $\mathcal{N}$ and $\mathcal{M}$ be labelled nets and A be a synchronisation set such that $\mathcal{N} \simeq \mathcal{M}\,\mathsf{sy}\,A$. Then $\mathcal{N} \cong \mathcal{N}\,\mathsf{sy}\,A$.

Proof. Let $\mathcal{P}=\mathcal{N}\,\mathsf{sy}\,A$ and $\mathcal{R}=\mathcal{M}\,\mathsf{sy}\,A$. By corollary 3.3, it suffices to show that for every transition $u\odot w \in T_{\mathcal{P}} - T_{\mathcal{N}}$, there is $t \in T_{\mathcal{N}}$ such that $t \simeq_{\mathcal{P}} u\odot w$. Suppose that $\mathcal{N} \simeq_g \mathcal{R}$. There are v and z such that $u \simeq_g v$ and $w \simeq_g z$. Clearly, $v\odot z \in T_{\mathcal{R}}$. Hence there is $t \in T_{\mathcal{N}}$ such that $t \simeq_g v\odot z$. By proposition 2.3(1) and $v\odot z \bowtie_{\mathcal{R}} vz$, we have $t \bowtie_{\mathcal{N}} uw$ which in turn means that $t \simeq_{\mathcal{P}} u\odot w$. □

The next proposition gathers together a number of simple facts involving synchronisation and duplication equivalence. Note that the second item implies that as far as duplication equivalence is concerned, synchronisation is idempotent and commutative.

Proposition 3.5 Let $\mathcal{N}$ be a labelled net, and A and B be synchronisation sets.

1. $\mathcal{N} \simeq \mathcal{N}\,\mathsf{sy}\,A$ if and only if $\mathcal{N} \cong \mathcal{N}\,\mathsf{sy}\,A$.

2. $\mathcal{N}\,\mathsf{sy}\,A\,\mathsf{sy}\,B \cong \mathcal{N}\,\mathsf{sy}\,(A \cup B)$.
3. If $\mathcal{N}\,\mathsf{sy}\,A \simeq \mathcal{N}\,\mathsf{sy}\,B$ then $\mathcal{N}\,\mathsf{sy}\,A \cong \mathcal{N}\,\mathsf{sy}\,B$.
4. If $A \subseteq B$ and $\mathcal{N} \simeq \mathcal{N}\,\mathsf{sy}\,B$ then $\mathcal{N} \cong \mathcal{N}\,\mathsf{sy}\,A$.
5. If $\mathcal{N} \simeq \mathcal{N}\,\mathsf{sy}\,A$ and $\mathcal{N} \simeq \mathcal{N}\,\mathsf{sy}\,B$ then $\mathcal{N} \cong \mathcal{N}\,\mathsf{sy}\,(A \cup B)$.

Proof. (1) Follows directly from proposition 3.4 and the definition of $\cong$.
(2) Follows directly from proposition 2.4(2) and the definition of synchronisation.
(3) By $\mathcal{N}\,\mathsf{sy}\,A \simeq \mathcal{N}\,\mathsf{sy}\,B$ and proposition 3.4, there are g and f such that

$$\mathcal{N}\,\mathsf{sy}\,A \ \cong_g\ \mathcal{N}\,\mathsf{sy}\,A\,\mathsf{sy}\,B \quad \text{and} \quad \mathcal{N}\,\mathsf{sy}\,B \ \cong_f\ \mathcal{N}\,\mathsf{sy}\,B\,\mathsf{sy}\,A.$$

Moreover, by proposition 3.5(2), there are d and e such that

$$\mathcal{N}\,\mathsf{sy}\,A\,\mathsf{sy}\,B \ \cong_d\ \mathcal{N}\,\mathsf{sy}\,(A \cup B) \quad \text{and} \quad \mathcal{N}\,\mathsf{sy}\,B\,\mathsf{sy}\,A \ \cong_e\ \mathcal{N}\,\mathsf{sy}\,(B \cup A).$$

Hence, by property (1) and $A \cup B = B \cup A$, $h = g \circ d \circ e^{-1} \circ f^{-1}$ is an isomorphism satisfying $\mathcal{N}\,\mathsf{sy}\,A \cong_h \mathcal{N}\,\mathsf{sy}\,B$.
(4) Follows from $\mathcal{N}\,\mathsf{sy}\,B = \mathcal{N}\,\mathsf{sy}\,(B - A)\,\mathsf{sy}\,A$ and proposition 3.4.
(5) From propositions 3.5(2) and 3.2, we obtain $\mathcal{N} \simeq \mathcal{N}\,\mathsf{sy}\,A \simeq \mathcal{N}\,\mathsf{sy}\,B\,\mathsf{sy}\,A \simeq \mathcal{N}\,\mathsf{sy}\,(A \cup B)$. Then we apply proposition 3.5(1). $\square$

Finally, we have that being a $\otimes$-set is preserved through synchronisation.

Proposition 3.6 If P is a $\otimes$-set for unionable labelled nets $\mathcal{N}$ and $\mathcal{M}$, and A, B are synchronisation sets then P is also a $\otimes$-set for $\mathcal{N}\,\mathsf{sy}\,A$ and $\mathcal{M}\,\mathsf{sy}\,B$.

Proof. Follows directly from the definition of $\otimes$-sets and proposition 3.1. $\square$

If one looks at the last item of proposition 3.5 then it is obvious that for every net $\mathcal{N}$ there exists the *maximal* [3] *synchronisation set* A such that $\mathcal{N} \simeq \mathcal{N}\,\mathsf{sy}\,A$. We will denote this set by $\mathsf{max}_{\mathcal{N}}$. Note that

$$\mathsf{max}_{\mathcal{N}} = \bigcup \{A \mid \mathcal{N} \simeq \mathcal{N}\,\mathsf{sy}\,A\}.$$

Hence, by proposition 3.4, if $\mathcal{N} \simeq \mathcal{M}$ then $\mathsf{max}_{\mathcal{N}} = \mathsf{max}_{\mathcal{M}}$.

3.1 Net union, duplication equivalence and synchronisation

Crucial to the further development is the way in which combining the net union and synchronisation operations affect duplication equivalence. The first result we obtain is that, from the point of view of duplication equivalence, synchronisation propagates through net union.

Proposition 3.7 Let $\mathcal{N}_1, \ldots, \mathcal{N}_k$ be pairwise unionable labelled nets and A be a synchronisation set. Then

$$(\mathcal{N}_1 \cup \ldots \cup \mathcal{N}_k)\,\mathsf{sy}\,A \quad \cong \quad ((\mathcal{N}_1\,\mathsf{sy}\,A) \cup \ldots \cup (\mathcal{N}_k\,\mathsf{sy}\,A))\,\mathsf{sy}\,A.$$

Proof. Let $\mathcal{N} = (\mathcal{N}_1 \cup \ldots \cup \mathcal{N}_k)\,\mathsf{sy}\,A$ and $\mathcal{M} = ((\mathcal{N}_1\,\mathsf{sy}\,A) \cup \ldots \cup (\mathcal{N}_k\,\mathsf{sy}\,A))\,\mathsf{sy}\,A$. Since $\mathcal{N}$ and $\mathcal{M}$ satisfy the condition in proposition 2.4(2), $\mathcal{N} \cong \mathcal{M}$. $\square$

We now formulate a key result on the relationship between net union, synchronisation and duplication equivalence.

[3] Maximal w.r.t. set inclusion.

Proposition 3.8 Let $\mathcal{P} = \mathcal{N}_1 \cup \ldots \cup \mathcal{N}_k$ where $\mathcal{N}_1, \ldots, \mathcal{N}_k$ are pairwise unionable nets, and let A and B be synchronisation sets such that for all distinct $i, j \leq k$, the following hold.

- $\mathcal{N}_i \,\mathsf{sy}\, A \simeq \mathcal{N}_i \,\mathsf{sy}\, B$.
- For all $t \in T_i$ and $v \in T_j$, if $(t, v) \in \mathsf{syn}_A$ (or $(t, v) \in \mathsf{syn}_B$) then there is $w \in T_\mathcal{P}$ such that $\lambda_\mathcal{P}(w) = \imath$ and $tv \bowtie_\mathcal{P} w$, or there are $u, w \in T_\mathcal{P}$ such that $(u, w) \in \mathsf{syn}_B$ (resp. $(u, w) \in \mathsf{syn}_A$) and $tv \bowtie_\mathcal{P} uw$.

Then $\mathcal{P} \,\mathsf{sy}\, A \cong \mathcal{P} \,\mathsf{sy}\, B$.

Proof. Let $\mathcal{N} = \mathcal{P} \,\mathsf{sy}\, A$ and $\mathcal{M} = \mathcal{P} \,\mathsf{sy}\, B$. From propositions 3.5(3) and 2.4(1) it follows that we can apply proposition 2.4(2). Hence $\mathcal{N} \cong \mathcal{M}$. □

Later, we will need a specific instance of the last result for $B = A \cup \mathsf{a}$.

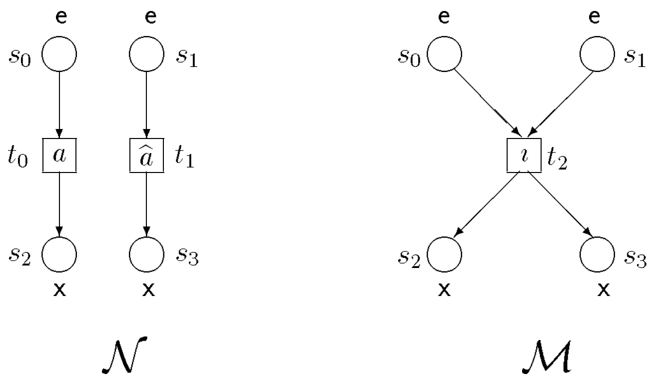
Corollary 3.9 Let $\mathcal{P} = \mathcal{N}_1 \cup \ldots \cup \mathcal{N}_k$ where $\mathcal{N}_1, \ldots, \mathcal{N}_k$ are pairwise unionable nets, and let A be a synchronisation set and $a \notin A$ be a communication action such that for all distinct $i, j \leq k$, the following hold.

- $\mathcal{N}_i \,\mathsf{sy}\, (A \cup \mathsf{a}) \simeq \mathcal{N}_i \,\mathsf{sy}\, A$.
- For all $(t, v) \in (T_i \times T_j) \cap \mathsf{syn}_\mathsf{a}$ there is $w \in T_\mathcal{P}$ such that $\lambda_\mathcal{P}(w) = \imath$ and $tv \bowtie_\mathcal{P} w$, or there are $u, w \in T_\mathcal{P}$ such that $(u, w) \in \mathsf{syn}_A$ and $tv \bowtie_\mathcal{P} uw$.

Then $\mathcal{P} \,\mathsf{sy}\, (A \cup \mathsf{a}) \cong \mathcal{P} \,\mathsf{sy}\, A$. □

3.1.1 Reversing Corollary 3.9

We will now aim at reversing corollary 3.9 for $k = 2$. We need such a result to characterise maximal synchronisation sets of nets which are obtained through composition. In particular, we would like to show that if $(\mathcal{N} \cup \mathcal{M}) \,\mathsf{sy}\, (A \cup \mathsf{a}) \simeq (\mathcal{N} \cup \mathcal{M}) \,\mathsf{sy}\, A$ then $\mathcal{N} \,\mathsf{sy}\, (A \cup \mathsf{a}) \simeq \mathcal{N} \,\mathsf{sy}\, A$. However, such a result does not, in general, hold. For consider the following two nets (note that the union of the two nets in this example corresponds to the box expression $(a \| \widehat{a}) \,\square\, \imath)$.



Then, clearly, $(\mathcal{N} \cup \mathcal{M}) \,\mathsf{sy}\, \mathsf{a} \simeq \mathcal{N} \cup \mathcal{M}$ but $\mathcal{N} \,\mathsf{sy}\, \mathsf{a} \not\simeq \mathcal{N}$. Intuitively, the reason why we have a problem here is that, in $\mathcal{N} \cup \mathcal{M}$, the transition t_2 can be interpreted both as coming from $\mathcal{M}$, and as coming from $\mathcal{N}$ (e.g. by being the 'result' of synchronising transitions t_0 and t_1). We therefore need first to constrain the two nets, $\mathcal{N}$ and $\mathcal{M}$, before trying to reverse the corollary. To this end, suppose that $\mathcal{N}$ and $\mathcal{M}$ are two unionable labelled nets, A is a synchronisation set, and $a \notin A$ is a communication action. We also make six specific assumptions, A1-A6 below.

A1 In $\mathcal{N}$ and $\mathcal{M}$, all the transitions labelled with communication actions are simple, and every $\imath$-labelled transition t satisfies $W(s,t) \leq 2$ and $W(t,s) \leq 2$, for every place s.

A2 There exist two disjoint non-empty sets of places, E and X, of $\mathcal{N}$ such that for every transition t in $\mathcal{N}$, $t^\bullet \cap E = {}^\bullet t \cap X = \emptyset$.

A3 If t is a transition in $\mathcal{N}$ labelled with a communication action then ${}^\bullet t \cap E = \emptyset \vee {}^\bullet t \subseteq E$ and $t^\bullet \cap X = \emptyset \vee t^\bullet \subseteq X$.

A4 The set of common places of $\mathcal{N}$ and $\mathcal{M}$ is either $\emptyset$ or E or X or $E \cup X$.

A5 If E (or X) is included in $S_\mathcal{N} \cap S_\mathcal{M}$ then E (resp. X) is a $\otimes$-set for $\mathcal{N}$ and $\mathcal{M}$.

We call a transition t in $(\mathcal{N} \cup \mathcal{M})\,\mathsf{sy}\,(A \cup \mathsf{a})$ an EX*-transition*, and denote $t \in T_{EX}$, if ${}^\bullet t = E$, $t^\bullet = X$ and $(t,E),(t,X) \in \mathsf{const}_{(\mathcal{N}\cup\mathcal{M})\,\mathsf{sy}\,(A\cup\mathsf{a})}$.

A6 If t is an EX-transition in $(\mathcal{N} \cup \mathcal{M})\,\mathsf{sy}\,A$ whose label belongs to $A \cup \{\imath\}$, then there is $u \in T_{\mathcal{N}\,\mathsf{sy}\,A}$ such that $t \simeq_{(\mathcal{N}\cup\mathcal{M})\,\mathsf{sy}\,A} u$.

Intuitively, E and/or X form the interface between $\mathcal{N}$ and $\mathcal{M}$, E being derived from the entry, and X being derived from the exit places of $\mathcal{N}$ (later, the sets E and X will be derived through the place multiplication and addition operators, which will, in particular, guarantee that A5 holds). In the last example, $E = \{s_0, s_1\}$ and $X = \{s_2, s_3\}$. Note also that this example fails to satisfy A6 for $t = t_2$. It is worth mentioning that A1-A5 are conditions which will be satisfied by all the nets associated with box expressions or being used in the definitions of the composition operators (see proposition 5.1). However, the nature of condition A6 is different, as our last example has demonstrated. The next result can be seen a reverse of the first part of corollary 3.9.

Proposition 3.10 If $(\mathcal{N} \cup \mathcal{M})\,\mathsf{sy}\,(A \cup \mathsf{a}) \simeq (\mathcal{N} \cup \mathcal{M})\,\mathsf{sy}\,A$ then $\mathcal{N}\,\mathsf{sy}\,(A \cup \mathsf{a}) \simeq \mathcal{N}\,\mathsf{sy}\,A$.

Proof. By corollary 3.3 and proposition 3.5(2), it suffices to show that if $t \in T_{\mathcal{N}\,\mathsf{sy}\,(A\cup\mathsf{a})} - T_{\mathcal{N}\,\mathsf{sy}\,A}$ then $t \simeq_{\mathcal{N}\,\mathsf{sy}\,(A\cup\mathsf{a})} w$ for some transition w in $\mathcal{N}\,\mathsf{sy}\,A$. From $(\mathcal{N} \cup \mathcal{M})\,\mathsf{sy}\,(A \cup \mathsf{a}) \simeq (\mathcal{N} \cup \mathcal{M})\,\mathsf{sy}\,A$, corollary 3.3 and proposition 3.5(2), it follows that there is u in $(\mathcal{N} \cup \mathcal{M})\,\mathsf{sy}\,A$ such that $t \simeq_{(\mathcal{N}\cup\mathcal{M})\,\mathsf{sy}\,(A\cup\mathsf{a})} u$. If $u \in T_{\mathcal{N}\,\mathsf{sy}\,A}$ then we can take $w = u$. If $u \in T_{\mathcal{M}\,\mathsf{sy}\,A}$ then, from propositions 2.2 and 3.6, A2, A4 and A5, it follows that $u \in T_{EX}$. Hence, by A6, there is $w \in T_{\mathcal{N}\,\mathsf{sy}\,A}$ such that $w \simeq_{(\mathcal{N}\cup\mathcal{M})\,\mathsf{sy}\,A} u$. Thus $t \simeq_{\mathcal{N}\,\mathsf{sy}\,(A\cup\mathsf{a})} w$. Finally, assume that $u = v \odot z$ where $v \in T_\mathcal{N}$ and $z \in T_\mathcal{M}$. Then, again from propositions 2.2 and 3.6, A2, A4 and A5, it follows that $z \in T_{EX}$. Hence, by A6 (note that $a \notin A$), there is $y \in T_\mathcal{N}$ such that $z \simeq_{(\mathcal{N}\cup\mathcal{M})\,\mathsf{sy}\,A} y$. Thus $w = v \odot y$, which belongs to $\mathcal{N}\,\mathsf{sy}\,A$, satisfies $t \simeq_{\mathcal{N}\,\mathsf{sy}\,(A\cup\mathsf{a})} w$. □

We then partially reverse the second part of corollary 3.9.

Proposition 3.11 Let $(\mathcal{N} \cup \mathcal{M})\,\mathsf{sy}\,(A \cup \mathsf{a}) \simeq (\mathcal{N} \cup \mathcal{M})\,\mathsf{sy}\,A$ and $(t,v) \in (T_\mathcal{N} \times (T_\mathcal{M} - T_{EX})) \cap \mathsf{syn}_\mathsf{a}$. Moreover, whenever $t \in T_{EX}$, there is no $x \in T_{\mathcal{M}\,\mathsf{sy}\,A}$ such that $t \odot v \simeq_{(\mathcal{N}\cup\mathcal{M})\,\mathsf{sy}\,(A\cup\mathsf{a})} x$. Then there are transitions $(u,w) \in (T_\mathcal{N} \times T_\mathcal{M}) \cap \mathsf{syn}_A$ such that $t \bowtie_\mathcal{N} u$ and $v \bowtie_\mathcal{M} w$.

Proof. By corollary 3.3 and proposition 3.5(2), there is an $x \in T_{(\mathcal{N}\cup\mathcal{M})\,\mathsf{sy}\,A}$ such that $t \odot v \simeq_{(\mathcal{N}\cup\mathcal{M})\,\mathsf{sy}\,(A\cup\mathsf{a})} x$. If $x \in T_{\mathcal{M}\,\mathsf{sy}\,A}$ (or $x \in T_{\mathcal{N}\,\mathsf{sy}\,A}$) then, from propositions 2.2 and 3.6, A1, A2, A4 and A5, it follows that $t \in T_{EX}$ (resp. $v \in T_{EX}$), a contradiction with the assumptions about t (resp. v) we made. Hence $x = u \odot w$ where $u \in T_\mathcal{N}$ and $w \in T_\mathcal{M}$. By proposition 2.4(1), $tv \bowtie_{\mathcal{N}\cup\mathcal{M}} uw$. What we need to show is that $t \bowtie_\mathcal{N} u$ and $v \bowtie_\mathcal{M} w$. By proposition 2.1, it suffices to show the latter.

Suppose $v \bowtie_\mathcal{M} w$ does not hold. Without loss of generality, and by A1, we may assume that $W_\mathcal{M}(s,v) = 1$ and $W_\mathcal{M}(s,w) = 0$, for some $s \in S_\mathcal{M}$. If $s \notin S_\mathcal{N} \cap S_\mathcal{M}$ then we obtain a contradiction with $tv \bowtie_{\mathcal{N}\cup\mathcal{M}} uw$. If $s \in S_\mathcal{N} \cap S_\mathcal{M}$ then, by $tv \bowtie_{\mathcal{N}\cup\mathcal{M}} uw$ and A1, we have $W_\mathcal{N}(s,t) = 0$ and

$W_{\mathcal{N}}(s,u)=1$. By $s \in {}^\bullet v$ and A2, $s \in E$. If ${}^\bullet t \cap E = \emptyset$ then, by A2, there is $q \in S_{\mathcal{N}} - (E \cup X)$ such that $W_{\mathcal{N}}(q,t)=1$. On the other hand, by A3 and $s \in {}^\bullet u$, $W_{\mathcal{N}}(q,u)=0$, contradicting $tv \bowtie_{\mathcal{N}\cup\mathcal{M}} uw$. Hence ${}^\bullet t \subseteq E$. As a result, there is $r \in E$ such that $W_{\mathcal{N}}(r,t)=1$. By A5, there is $p \in E$ such that $r \simeq_{\mathcal{N}} p$ and $s \simeq_{\mathcal{M}} p$. We now observe that

$$\begin{array}{rcccl} & W_{\mathcal{N}}(p,t) & + & W_{\mathcal{M}}(p,v) & = & 1+1 \\ \text{and} & W_{\mathcal{N}}(p,u) & + & W_{\mathcal{M}}(p,w) & \leq & 1+0 \end{array}$$

contradicting $tv \bowtie_{\mathcal{N}\cup\mathcal{M}} uw$. $\square$

4 Composition operators

The PBC employs net operators which aim at capturing common concurrent programming constructs. We have already introduced one such operator, i.e., synchronisation. The remaining four operators are the choice, parallel, sequence and iteration compositions, denoted respectively by $\mathcal{N}_1 \square \mathcal{N}_2$, $\mathcal{N}_1 \| \mathcal{N}_2$, $\mathcal{N}_1 ; \mathcal{N}_2$ and $[\mathcal{N}_1 * \mathcal{N}_2 * \mathcal{N}_3]$ where the entry and exit places of the component nets $\mathcal{N}_i$ are used to construct a composite net. The definitions of the four operators are preceded by three auxiliary notions, viz. place addition, place multiplication and gluing of nets.

We first formalise what it means to replace a place by a set of other places which inherit its connectivity. Let $\mathcal{N}$ be a labelled net and $s_1,\ldots,s_k$ be its places. Moreover, let $S_1,\ldots,S_k$ be disjoint non-empty sets of places not in $\mathcal{N}$ and $l_1,\ldots,l_k \in \{\mathsf{e},\mathsf{i},\mathsf{x}\}$. Then

$$\mathcal{N} \oplus \{(s_1,S_1,l_1),\ldots,(s_k,S_k,l_k)\} = (S',T',W',\lambda')$$

is a net such that $S' = S - \{s_1,\ldots,s_k\} \cup (S_1 \cup \ldots \cup S_k)$, $T' = T$ and, for all $n,m \in S' \cup T'$,

$$W'(n,m) \;=\; \begin{cases} W(n,m) & \text{if } n,m \in S \cup T \\ W(s_i,m) & \text{if } n \in S_i,\ m \in S \cup T \\ W(n,s_j) & \text{if } n \in S \cup T,\ m \in S_j \\ 0 & \text{if } n \in S_i,\ m \in S_j \end{cases} \qquad \text{and} \qquad \lambda'(n) \;=\; \begin{cases} \lambda(n) & \text{if } n \in S \cup T \\ l_i & \text{if } n \in S_i. \end{cases}$$

A *multiplication* of non-empty disjoint sets of places $S_1,\ldots,S_k$ $(k \geq 2)$ is a set of places

$$S_1 \otimes \ldots \otimes S_k \;=\; \{\{s_1,\ldots,s_k\} \mid s_1 \in S_1 \wedge \ldots \wedge s_k \in S_k\}.$$

In the above, each $\{s_1,\ldots,s_k\}$ is a place which is assumed 'fresh' and hence different from every place in the nets to which the places $S_1 \otimes \ldots \otimes S_k$ might be added using the place addition operator, $\oplus$.

Let $\mathcal{N}_1,\ldots,\mathcal{N}_k$ be disjoint labelled nets, and Δ be a *gluing set.* The latter is defined by:

$$\Delta \;=\; \{(S^1_1,\ldots,S^1_{r_i},l_1),\ldots(S^m_1,\ldots,S^1_{r_m},l_m)\}$$

where $m,r_1,\ldots,r_m \geq 1$ and $l_1,\ldots,l_m \in \{\mathsf{e},\mathsf{i},\mathsf{x}\}$ and each S^j_i is the set of entry places or the set of exit places of one of the nets $\mathcal{N}_1,\ldots,\mathcal{N}_k$. It is assumed that, for every $j \leq k$, both ${}^\circ\mathcal{N}_j$ and $\mathcal{N}_j{}^\circ$ can appear in Δ at most once and never in the same element of Δ. With such assumptions we define, for $j \leq k$,

$$\mathcal{N}_j : \Delta \;=\; \mathcal{N}_j \oplus \{(s,[S^i_1 \otimes \ldots \otimes S^i_{r_i}]_s,l_i) \mid s \in {}^\circ\mathcal{N}_j \cup \mathcal{N}_j{}^\circ \wedge\;\; s \in S^i_1 \cup \ldots \cup S^i_{r_i}\}$$

where $[S^i_1 \otimes \ldots \otimes S^i_{r_i}]_s = \{p \in S^i_1 \otimes \ldots \otimes S^i_{r_i} \mid s \in p\}$. Then the net

$$(\mathcal{N}_1,\ldots,\mathcal{N}_k) : \Delta \;=\; (\mathcal{N}_1 : \Delta) \cup \ldots \cup (\mathcal{N}_k : \Delta)$$

where $\cup$ denotes net union, is a *glued net* obtained from nets $\mathcal{N}_1,\ldots,\mathcal{N}_k$ using the gluing set Δ.

The composition operators are defined for a class of nets called pre-boxes. A *pre-box* is a labelled net $\mathcal{N}$ such that the following hold:

1. $\mathcal{N}^\circ \neq \emptyset \neq {}^\circ\mathcal{N}$.

2. $(\mathcal{N}^\circ)^\bullet = {}^\bullet({}^\circ\mathcal{N}) = \emptyset$.

3. All the transitions labelled with communication actions are simple, and every $\imath$-labelled transition t satisfies $W(s,t) \leq 2$ and $W(t,s) \leq 2$, for every place s.

Let $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$ be disjoint pre-boxes. The four composition operators are defined thus.

- Sequential $\quad \mathcal{N}_1;\mathcal{N}_2 = (\mathcal{N}_1,\mathcal{N}_2) : \{(\mathcal{N}_1{}^\circ, {}^\circ\mathcal{N}_2, \mathsf{i})\}$.
- Choice $\quad \mathcal{N}_1 \,\square\, \mathcal{N}_2 = (\mathcal{N}_1,\mathcal{N}_2) : \{({}^\circ\mathcal{N}_1, {}^\circ\mathcal{N}_2, \mathsf{e}), (\mathcal{N}_1{}^\circ, \mathcal{N}_2{}^\circ, \mathsf{x})\}$.
- Concurrent $\quad \mathcal{N}_1 \| \mathcal{N}_2 = \mathcal{N}_1 \cup \mathcal{N}_2$.
- Iteration $\quad [\mathcal{N}_1 * \mathcal{N}_2 * \mathcal{N}_3] = (\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3,\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3) : \Delta$

where in the last case $\mathcal{M}_i$ is a disjoint copy of $\mathcal{N}_i$, for $i = 1,2,3$, and Δ is a gluing set given by

$$\Delta \;=\; \{({}^\circ\mathcal{N}_1, {}^\circ\mathcal{M}_1, \mathsf{e}), (\mathcal{N}_1{}^\circ, {}^\circ\mathcal{N}_2, \mathcal{M}_2{}^\circ, {}^\circ\mathcal{N}_3, \mathsf{i}), (\mathcal{M}_1{}^\circ, \mathcal{N}_2{}^\circ, {}^\circ\mathcal{M}_2, {}^\circ\mathcal{M}_3, \mathsf{i}), (\mathcal{N}_3{}^\circ, \mathcal{M}_3{}^\circ, \mathsf{x})\}.$$

Intuitively, sequential composition joins together the exit places of $\mathcal{N}_1$ with the entry places of $\mathcal{N}_2$; choice composition joins together respectively the entry and exist places of the two nets; and parallel composition simply juxtaposes the two nets. The iteration composition $[\mathcal{N}_1 * \mathcal{N}_2 * \mathcal{N}_3]$ is more complicated and its behaviour should be understood as follows. The net $\mathcal{N}_1$ represents the entry part of the iteration construct, after its completion $\mathcal{N}_2$ can be executed zero or more times, and after a successful completion of each execution of $\mathcal{N}_2$ one can pass the control to $\mathcal{N}_3$ which is the exit part of the composite nets. Moreover, in order to ensure that iteration always produces a 1-safe net provided that the $\mathcal{N}_i$ nets are 1-safe (see [2] for a detailed discussion), in the graph-theoretic construction of $[\mathcal{N}_1 * \mathcal{N}_2 * \mathcal{N}_3]$ each of the nets $\mathcal{N}_i$ appears twice, i.e., as $\mathcal{N}_i$ and $\mathcal{M}_i$. The latter net plays exactly the same role as $\mathcal{N}_i$ as far as the behaviour of the composite net is concerned. Figure 5 illustrates the use of the four composition operators (this example also illustrates the definition of boxes in the first paragraph of the next section).

The sets of places resulting from place multiplication in the definition of the three composition operators can easily be seen as $\otimes$-sets of the composed nets. Consider, for example, the sequential composition

$$\mathcal{N}_1;\mathcal{N}_2 \;=\; \mathcal{N}_1';\mathcal{N}_2' \;=\; \mathcal{N}_1 : \{(\mathcal{N}_1{}^\circ, {}^\circ\mathcal{N}_2, \mathsf{i})\} \,\cup\, \mathcal{N}_2 : \{(\mathcal{N}_1{}^\circ, {}^\circ\mathcal{N}_2, \mathsf{i})\}.$$

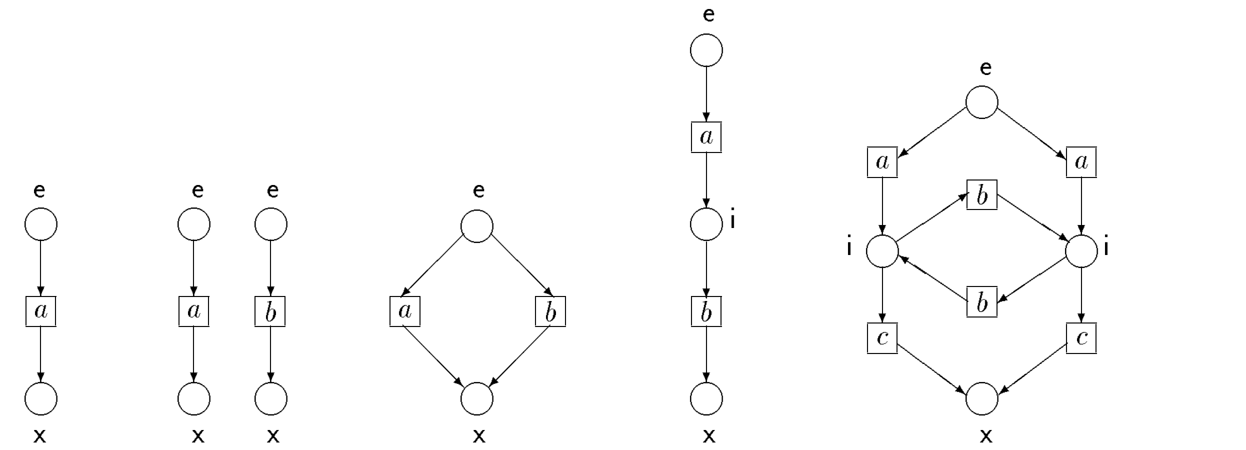
Then $\mathcal{N}_1{}^\circ \otimes {}^\circ\mathcal{N}_2$ is a $\otimes$-set for the nets $\mathcal{N}_1'$ and $\mathcal{N}_2'$. Indeed, if we take $s, r \in \mathcal{N}_1{}^\circ \otimes {}^\circ\mathcal{N}_2$ then $s = \{s_1, s_2\}$ and $r = \{r_1, r_2\}$, where $s_1, r_1 \in \mathcal{N}_1{}^\circ$ and $s_2, r_2 \in {}^\circ\mathcal{N}_2$. Then $p = \{s_1, r_2\}$ also belongs to $\mathcal{N}_1{}^\circ \otimes {}^\circ\mathcal{N}_2$ and satisfies $s \simeq_{\mathcal{N}_1'} p$ and $r \simeq_{\mathcal{N}_2'} p$.

5 Boxes

We now bring to our discussion a class of process expressions. In this paper we will deal with a subset of the Petri Box Calculus [2] assuming the following syntax of *box expressions*:

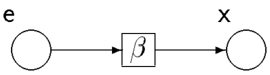
$$E \;::=\; \beta \;\mid\; E\,\mathsf{sy}\,A \;\mid\; E;E \;\mid\; E \,\square\, E \;\mid\; E\|E \;\mid\; [E * E * E] \tag{2}$$

In the above, β is an action in $\mathcal{A} \cup \{\imath\}$ and A is a synchronisation set. The five operators correspond to those introduced for labelled nets. There is a mapping which associates with every box expression, E, a labelled net, $\mathsf{box}(E)$, in the following way:



$\mathsf{box}(a)$ $\mathsf{box}(a\|b) = \mathcal{N}_a\|\mathcal{N}_b$ $\mathsf{box}(a \Box b) = \mathcal{N}_a \Box \mathcal{N}_b$ $\mathsf{box}(a;b) = \mathcal{N}_a;\mathcal{N}_b$ $\mathsf{box}([a*b*c]) = [\mathcal{N}_a * \mathcal{N}_b * \mathcal{N}_c]$

Figure 5: Examples of boxes where $\mathcal{N}_x = \mathsf{box}(x)$ for $x \in \{a,b,c\}$.

$\mathsf{box}(\beta) =$ 

$$
\begin{aligned}
\mathsf{box}(E\,\mathsf{sy}\,A) &= \mathsf{box}(E)\,\mathsf{sy}\,A\\
\mathsf{box}(E;F) &= \mathsf{box}(E);\mathsf{box}(F)\\
\mathsf{box}(E \Box F) &= \mathsf{box}(E) \Box \mathsf{box}(F)\\
\mathsf{box}(E\|F) &= \mathsf{box}(E)\|\mathsf{box}(F)\\
\mathsf{box}([E*F*G]) &= [\mathsf{box}(E)*\mathsf{box}(F)*\mathsf{box}(G)].
\end{aligned}
$$

In what follows, we will call a *box* a net which can be derived from a box expression through the $\mathsf{box}()$ mapping. In general, isomorphic boxes will be identified. The following collection of simple facts about boxes can easily be proven by induction on the structure of expressions generating them.

Proposition 5.1 Let $\mathcal{B}$ be a box, t be its transition and s be its place.

1. $\mathcal{B}$ is a pre-box (and so also a labelled net).
2. If t is labelled with a communication action then
   (a) ${}^\bullet t \cap {}^\circ\mathcal{B} \neq \emptyset$ implies ${}^\bullet t \subseteq {}^\circ\mathcal{B}$,
   (b) $t^\bullet \cap \mathcal{B}^\circ \neq \emptyset$ implies $t^\bullet \subseteq \mathcal{B}^\circ$, and
   (c) ${}^\bullet t \subseteq {}^\circ\mathcal{B}$ and $t^\bullet \subseteq \mathcal{B}^\circ$ together imply that ${}^\bullet t = {}^\circ\mathcal{B}$ if and only if $t^\bullet = \mathcal{B}^\circ$.
3. s has no duplicates other than itself, and if ${}^\bullet s = \emptyset$ (or $s^\bullet = \emptyset$) then $s \in {}^\circ\mathcal{B}$ (resp. $s \in \mathcal{B}^\circ$).
4. If t is not a simple transition, or if ${}^\circ\mathcal{B} \cup \mathcal{B}^\circ$ is a proper subset of ${}^\bullet t \cup t^\bullet$ then t is labelled by $\imath$ and there are transitions u and w in $\mathcal{B}$ with conjugate labels such that $t \bowtie_{\mathcal{B}} uw$. $\Box$

5.1 Duplication equivalent boxes

Our first goal (crucial from the point of view of dealing with axiomatisation of duplication equivalence of box expressions) is to structurally characterise the maximal synchronisation sets of boxes. To this end we introduce some auxiliary sets of transitions, called ex-transitions and choice context transitions. An *ex–transition* of a box $\mathcal{B}$ is a simple transition t such that ${}^\bullet t = {}^\circ\mathcal{B}$ and $t^\bullet = \mathcal{B}^\circ$. We denote this by $t \in \mathsf{EX}_{\mathcal{B}}$ and use $\mathsf{ex}_{\mathcal{B}}$ to denote the set of labels of all ex–transitions of $\mathcal{B}$.[4] Choice context transitions duplicate each other except that they may have different (communication) labels. The terminology is motivated by the fact that such transitions always result from applying the choice composition operator. A set of *choice context* transitions of a box $\mathcal{B}$ is a maximal non-empty set U of transitions labelled with communication actions and all having the same connectivity. We also denote $\mathsf{ccall}_{\mathcal{B}} = \{\lambda_{\mathcal{B}}(U) \mid U \text{ is a set of choice context transitions}\}$. If the transitions in U are not ex-transitions then U is a set of *internal* choice context transitions and we define $\mathsf{ccint}_{\mathcal{B}} = \{\lambda_{\mathcal{B}}(U) \mid U \text{ is a set of internal choice context transitions}\}$. Note that $\mathsf{ex}_{\mathcal{B}}$ is a set of actions, and $\mathsf{ccall}_{\mathcal{B}}$ and $\mathsf{ccint}_{\mathcal{B}}$ are sets of sets of communication actions. For the boxes in figure 6, we have $\mathsf{ex}_{\mathcal{B}} = \{a\}$, $\mathsf{ccall}_{\mathcal{B}} = \{\{\widehat{a}, \widehat{b}\}, \{\widehat{c}\}, \{a\}\}$, $\mathsf{ccint}_{\mathcal{B}} = \{\{\widehat{a}, \widehat{b}\}, \{\widehat{c}\}\}$, $\mathsf{ex}_{\mathcal{C}} = \{a, b, c, \imath\}$, $\mathsf{ccall}_{\mathcal{C}} = \{\{a, b, c\}\}$ and $\mathsf{ccint}_{\mathcal{C}} = \emptyset$.

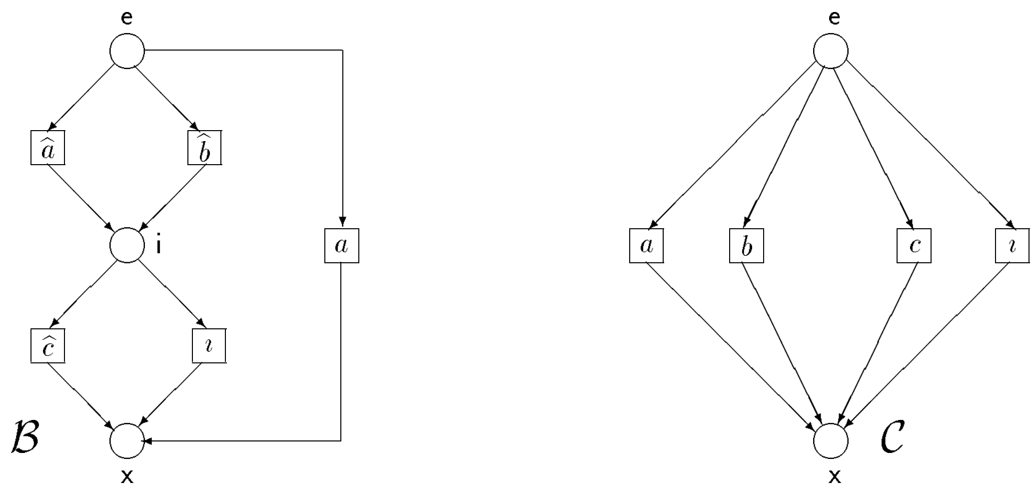


Figure 6: Boxes generated by $((\widehat{a} \square \widehat{b}); (\widehat{c} \square \imath)) \square a$ and $(a \square b) \square (c \square \imath)$.

Proposition 5.2 Let $\mathcal{B}$ be a box and t be its transition.

1. If $t \bowtie_{\mathcal{B}} u\delta$ for some transition u in $\mathcal{B}$ labelled by a communication action, then $\mathsf{EX}_{\mathcal{B}} \neq \emptyset$.
2. If ${}^\bullet t = {}^\circ\mathcal{B}$, $t^\bullet = \mathcal{B}^\circ$, $(t, {}^\circ\mathcal{B}) \in \mathsf{const}_{\mathcal{B}}$, $(t, \mathcal{B}^\circ) \in \mathsf{const}_{\mathcal{B}}$ and t is not an ex-transition then there are two ex-transitions u and w in $\mathcal{B}$ with conjugate labels satisfying $t \bowtie_{\mathcal{B}} uw$.

Proof. (1) The result will follow if we can show that, for every $\mathcal{B}$ obtained from a synchronisation-free expression, the following hold: (i) it is impossible that $t \bowtie_{\mathcal{B}} u\delta$; and (ii) if v, w are transitions such that $vw \bowtie_{\mathcal{B}} u\delta$ then v or w or u is an ex-transition.

[4]The notion of an ex-transition as well as other auxiliary notation introduced in this section are defined for boxes; however, they extend without any change to labelled nets.

We observe that (i) follows from the fact that all transitions in $\mathcal{B}$ derived from a synchronisation-free expression are simple. Thus $t \bowtie_{\mathcal{B}} u\delta$ would imply, also due to proposition 5.1(2a,2b),[5] ${}^\bullet u = u^\bullet = \emptyset$, a contradiction.

Part (ii) can be proved by induction on the structure of the synchronisation-free expression E generating $\mathcal{B}$. In the cases when $E = F;G$ or $E = F\|G$ or $E = [F*G*H]$, one can conclude that $vw \bowtie_{\mathcal{B}} u\delta$ is simply impossible to satisfy. Suppose $E = F \,\square\, G$. We consider two cases:

Case 1: $v \in \mathsf{box}(F)$ and $w \in \mathsf{box}(G)$. Also, without loss of generality, let $u \in \mathsf{box}(F)$. Consider a slightly modified expression, $E' = (F \,\square\, \imath) \,\square\, G$. Then we have $vw \bowtie_{\mathsf{box}(E')} uz$ where z is the only transition of $\mathsf{box}(\imath)$. Hence, by proposition 2.2, w is an ex-transition in $\mathsf{box}(G)$ and so also in $\mathcal{B}$.

Case 2: $v, w \in \mathsf{box}(F)$. If $u \in \mathsf{box}(F)$ then we have $vw \bowtie_{\mathsf{box}(F)} u\delta$ and can use the induction hypothesis. If $u \in \mathsf{box}(G)$ then we consider the same E' as in the first case to conclude that u is an ex-transition in $\mathcal{B}$.

(2) Since t is not a simple transition, it must have arisen as a synchronisation of two transitions u and w with conjugate labels and satisfying ${}^\bullet u \cup {}^\bullet w \subseteq {}^\circ\mathcal{B}$ and $u^\bullet \cup w^\bullet \subseteq \mathcal{B}^\circ$. Moreover, by $(t, {}^\circ\mathcal{B}) \in \mathsf{const}_{\mathcal{B}}$ and $(t, \mathcal{B}^\circ) \in \mathsf{const}_{\mathcal{B}}$, ${}^\bullet u = {}^\bullet w = {}^\circ\mathcal{B}$ and $u^\bullet = w^\bullet = \mathcal{B}^\circ$. Hence u and w are ex-transitions. □

The basic idea behind the structural characterisation of maximal synchronisation sets is that one can apply an a-synchronisation, without losing duplication equivalence, if for every pair of a–synchronisable transitions t and u it is possible to find a duplicate of their synchronisation in at least one of two different ways: as a syntactically generated $\imath$-transition, or as a synchronisation of two transitions with the same connectivity as t and u. To make the latter point explicit, suppose that $(\mathcal{B}\|\mathcal{C})\,\mathsf{sy}\,A \simeq (\mathcal{B}\|\mathcal{C})\,\mathsf{sy}\,A\,\mathsf{sy}\,\mathsf{a}$. Then, if t is a transition in $\mathcal{B}$ and u is a transition in $\mathcal{C}$ then we must be able to find A-synchronisable transitions, t' in $\mathcal{B}$ and u' in $\mathcal{C}$, with the same connectivity as respectively t and u. In other words, a necessary condition for $(\mathcal{B}\|\mathcal{C})\,\mathsf{sy}\,A \simeq (\mathcal{B}\|\mathcal{C})\,\mathsf{sy}\,A\,\mathsf{sy}\,\mathsf{a}$ to hold is that for every pair of a–synchronisable transitions t and u from respectively $\mathcal{B}$ and $\mathcal{C}$, there is a pair of A–synchronisable transitions, t' and u', which have the same connectivity as respectively t and u. We can express this rather conveniently using the sets $\mathsf{ccall}_{\mathcal{B}}$ and $\mathsf{ccall}_{\mathcal{C}}$ and some auxiliary notation. Let $\mathcal{Z}$ and $\mathcal{W}$ be two sets of sets of actions and A be a synchronisation set. Then $\mathsf{cov}^A(Z, W)$ is the set of all communication actions a such that if $Z \in \mathcal{Z}$ and $W \in \mathcal{W}$ satisfy $a \in Z \wedge \widehat{a} \in W$ or $\widehat{a} \in Z \wedge a \in W$ then there is $c \in A$ such that $c \in Z \wedge \widehat{c} \in W$. Define, for a synchronisation set A and boxes $\mathcal{B}, \mathcal{C}$,

$$\mathsf{covall}^A_{\mathcal{B}\mathcal{C}} \quad = \quad \mathsf{cov}^A(\mathsf{ccall}_{\mathcal{B}}, \mathsf{ccall}_{\mathcal{C}}).$$

We then observe that the above necessary condition for $(\mathcal{B}\|\mathcal{C})\,\mathsf{sy}\,A \simeq (\mathcal{B}\|\mathcal{C})\,\mathsf{sy}\,A\,\mathsf{sy}\,\mathsf{a}$ simply amounts to saying that a and $\widehat{a}$ belong to $\mathsf{covall}^A_{\mathcal{B}\mathcal{C}}$. Note that for the nets in figure 6, $\mathsf{covall}^{\mathsf{a}}_{\mathcal{B}\mathcal{C}} = \mathcal{A} - \mathsf{c}$. Characterising maximal synchronisation sets is rather straightforward in the case of sequential, parallel and iteration composition.

Proposition 5.3 Let $\mathcal{B}_i$ $(i = 1, 2, 3)$ be boxes and A be a synchronisation set. Then

$$\begin{array}{rcll} \mathsf{max}_{(\mathcal{B}_1;\mathcal{B}_2)\,\mathsf{sy}\,A} & = & \mathsf{covall}^A_{\mathcal{B}_1\mathcal{B}_2} & \cap \ \displaystyle\bigcap_{i=1}^{2} \mathsf{max}_{\mathcal{B}_i\,\mathsf{sy}\,A} \\ \mathsf{max}_{(\mathcal{B}_1\|\mathcal{B}_2)\,\mathsf{sy}\,A} & = & \mathsf{covall}^A_{\mathcal{B}_1\mathcal{B}_2} & \cap \ \displaystyle\bigcap_{i=1}^{2} \mathsf{max}_{\mathcal{B}_i\,\mathsf{sy}\,A} \\ \text{and} \quad \mathsf{max}_{[\mathcal{B}_1*\mathcal{B}_2*\mathcal{B}_3]\,\mathsf{sy}\,A} & = & \mathsf{covall}^A_{\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3} & \cap \ \displaystyle\bigcap_{i=1}^{3} \mathsf{max}_{\mathcal{B}_i\,\mathsf{sy}\,A}. \end{array}$$

Proof. We only consider the most complicated case, viz. iteration. To show the ($\subseteq$) inclusion, suppose that $a \notin A$ (clearly, if $a \in A$ then $a \in \mathsf{covall}^A_{\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3}$ and $a \in \mathsf{max}_{\mathcal{B}_i\,\mathsf{sy}\,A}$, for $i = 1, 2, 3$) and

$$[\mathcal{B}_1 * \mathcal{B}_2 * \mathcal{B}_3]\,\mathsf{sy}\,(A \cup \mathsf{a}) \ \simeq \ [\mathcal{B}_1 * \mathcal{B}_2 * \mathcal{B}_3]\,\mathsf{sy}\,A.$$

[5] Note that if $\mathcal{B}$ is derived from a synchronisation-free expression, then in the formulation of proposition 5.1(2) it does not matter whether t is labelled by a communication or internal action.

Let $\mathcal{N} = \mathcal{B}_1 : \{({}^\circ\mathcal{B}_1, {}^\circ\mathcal{D}_1, \mathsf{e}), (\mathcal{B}_1{}^\circ, {}^\circ\mathcal{B}_2, \mathcal{D}_2{}^\circ, {}^\circ\mathcal{B}_3, \mathsf{i})\}$ where each $\mathcal{D}_i$ is a copy of $\mathcal{B}_i$ (in the same way as $\mathcal{M}_i$ was a copy of $\mathcal{N}_i$ in the definition of the iteration operator). Moreover, let $\mathcal{M} = (\mathcal{B}_2, \mathcal{B}_3, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3) : \Delta$ be the union of all the remaining nets in the definition of $[\mathcal{B}_1 * \mathcal{B}_2 * \mathcal{B}_3]$. We will show that $a \in \mathsf{max}_{\mathcal{B}_i\, \mathsf{sy}\, A}$, for $i = 1, 2, 3$, and $a \in \mathsf{covall}^A_{\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3}$ using the results and notation from section 3.1.1.

Let $E = {}^\circ\mathcal{B}_1 \otimes {}^\circ\mathcal{D}_1$ and $X = \mathcal{B}_1{}^\circ \otimes {}^\circ\mathcal{B}_2 \otimes \mathcal{D}_2{}^\circ \otimes {}^\circ\mathcal{B}_3$. We observe that A1-A5 hold by the definition of the iteration operator and proposition 5.1(1,2). To show that A6 also holds, suppose that w is an EX-transition in $(\mathcal{N} \cup \mathcal{M})\, \mathsf{sy}\, A$. It is easy to see that then w belongs to $\mathcal{N}\, \mathsf{sy}\, A$ because every transition in $\mathcal{M}$ is connected to at least one place not in $E \cup X$.

Since A1-A6 hold, by proposition 3.10, we have $a \in \mathsf{max}_{\mathcal{N}\, \mathsf{sy}\, A}$. Moreover, $\mathcal{N} \simeq \mathcal{B}$, where $\mathcal{B}$ is $\mathcal{B}_1$ with the label of each exit place changed to i. Hence $a \in \mathsf{max}_{\mathcal{B}\, \mathsf{sy}\, A}$. We then observe that, by proposition 5.1(3), $[\mathcal{B}]_\simeq$ is $[\mathcal{B}_1]_\simeq$ with the label of every exit place changed to i. Hence $a \in \mathsf{max}_{\mathcal{B}_1\, \mathsf{sy}\, A}$. In a similar way, one may show that $a \in \mathsf{max}_{\mathcal{B}_2\, \mathsf{sy}\, A}$ and $a \in \mathsf{max}_{\mathcal{B}_3\, \mathsf{sy}\, A}$.

To show that $a \in \mathsf{covall}^A_{\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3}$ suppose that $(t, v) \in (T_\mathcal{N} \times T_\mathcal{M}) \cap \mathsf{syn}_\mathsf{a}$. It suffices to show that there are $(u, w) \in (T_\mathcal{N} \times T_\mathcal{M}) \cap \mathsf{syn}_A$ such that $t \bowtie_\mathcal{N} u$ and $v \bowtie_\mathcal{M} w$ (note that the operations of place multiplication and addition do not affect the relationship of having the same connectivity, and that by construction of the iteration operator it is not possible for two transitions coming from different nets forming $\mathcal{M}$ to have the same connectivity in $\mathcal{M}$). To be able to apply proposition 3.11, we first observe that $v \notin T_{EX}$ since no transition in $\mathcal{M}$ is an EX-transition. Suppose then that $t \in T_{EX}$ and $y, z \in T_\mathcal{M}$ are such that $tv \bowtie_{\mathcal{N}\cup\mathcal{M}} yz$. Then, without loss of generality, $y \in T_{\mathcal{D}_1}$ and $z \in T_{\mathcal{D}_2}$. Consequently, there are $s \in S_{\mathcal{D}_1} - {}^\circ\mathcal{D}_1$ and $r \in S_{\mathcal{D}_2} - \mathcal{D}_2{}^\circ$ such that $s \in {}^\bullet v$ and $p \in v^\bullet$, a contradiction. If we assume that $tv \bowtie_{\mathcal{N}\cup\mathcal{M}} y$ then a contradiction follows in a similar way. Hence there is no $x \in T_{\mathcal{M}\, \mathsf{sy}\, A}$ such that $t \odot v \simeq_{(\mathcal{N}\cup\mathcal{M})\, \mathsf{sy}\, (A \cup \mathsf{a})} x$. Hence proposition 3.11 can be applied.

Thus the $(\subseteq)$ inclusion holds. The reverse one follows from proposition 3.8. $\square$

Note that by setting $A = \emptyset$ we immediately obtain that, e.g., $\mathsf{max}_{\mathcal{B}_1;\mathcal{B}_2}$ is the set of all $a \in \mathsf{max}_{\mathcal{B}_1\, \mathsf{sy}\, A} \cap \mathsf{max}_{\mathcal{B}_2\, \mathsf{sy}\, A}$ such that if a transition labelled a or $\widehat{a}$ appears in $\mathcal{B}_1$ then there is no transition with the conjugate label in $\mathcal{B}_2$.

A similarly pleasant characterisation does not hold for the choice composition. One of the reasons is that a synchronisation of ex–transitions can sometimes be obtained by a syntactically introduced $\imath$-transition. For example, if $\mathcal{B} = \mathsf{box}(a \| \widehat{a})$ and $\mathcal{C} = \mathsf{box}(\imath)$ then we have $a \in \mathsf{max}_{(\mathcal{B} \,\Box\, \mathcal{C})\, \mathsf{sy}\, \emptyset}$ but $a \notin \mathsf{max}_{\mathcal{B}\, \mathsf{sy}\, \emptyset}$. Another example is provided by the boxes $\mathcal{B}$ and $\mathcal{C}$ in figure 6 for which we have $a \in \mathsf{max}_{(\mathcal{B} \,\Box\, \mathcal{C})\, \mathsf{sy}\, \mathsf{b}}$ but $a \notin \mathsf{max}_{\mathcal{B}\, \mathsf{sy}\, \mathsf{b}} = \mathsf{max}_\mathcal{B}$. Note that if we were to repeat our previous discussion for $(\mathcal{B} \,\Box\, \mathcal{C})\, \mathsf{sy}\, A \simeq (\mathcal{B} \,\Box\, \mathcal{C})\, \mathsf{sy}\, A\, \mathsf{sy}\, \mathsf{a}$ then it would no longer be true that t' and u' had to have the same connectivity as t and u if, e.g., u is an ex–transition in $\mathcal{C}$ since in such a case an ex–transition u' in $\mathcal{B}$ could provide a suitable 'match' for t'.

The characterisation of the maximal synchronisation sets for the choice composition is more complicated. For a box $\mathcal{B}$, let $\mathcal{U}$ be the set of all sets of internal choice context transitions U such that if $t \in U$ then there is no transition u in $\mathcal{B}$ satisfying $u \bowtie_\mathcal{B} \delta t$. Intuitively, this means that if we were to synchronise t with a conjugate ex-transition coming from the box $\mathcal{C}$ in the context $\mathcal{B} \,\Box\, \mathcal{C}$ then the resulting transition would not have the same connectivity as any of the transitions present in $\mathcal{B}$. We then define $\mathsf{ccnoex}_\mathcal{B} = \{\lambda_\mathcal{B}(U) \mid U \in \mathcal{U}\}$ and, for all boxes $\mathcal{B}$ and $\mathcal{C}$ and a synchronisation set A,

$$
\begin{array}{rcll}
\mathsf{covnoex}^A_{\mathcal{BC}} & = & \mathsf{cov}^A(\mathsf{ccnoex}_{\mathcal{B}\, \mathsf{sy}\, A}, \{\mathsf{ex}_\mathcal{C}\}) & \cap \;\; \mathsf{cov}^A(\mathsf{ccnoex}_{\mathcal{C}\, \mathsf{sy}\, A}, \{\mathsf{ex}_\mathcal{B}\}) \\
\mathsf{covmix}^A_{\mathcal{BC}} & = & \mathsf{cov}^A(\mathsf{ccint}_\mathcal{B}, \mathsf{ccall}_\mathcal{C}) & \cap \;\; \mathsf{cov}^A(\mathsf{ccint}_\mathcal{C}, \mathsf{ccall}_\mathcal{B}) \\
\text{and} \quad \mathsf{covint}^A_{\mathcal{BC}} & = & \mathsf{cov}^A(\mathsf{ccint}_\mathcal{B}, \mathsf{ccint}_\mathcal{C}). &
\end{array}
$$

The above definitions closely follow that of covall. For the nets in figure 6, we have $\mathsf{ccnoex}_\mathcal{B} = \{\{\widehat{a}, \widehat{b}\}, \{\widehat{c}\}\}$, $\mathsf{covnoex}^\mathsf{a}_{\mathcal{BC}} = \mathcal{A} - \mathsf{c}$, $\mathsf{covmix}^\mathsf{b}_{\mathcal{BC}} = \mathcal{A} - \mathsf{c}$ and $\mathsf{covint}^\emptyset_{\mathcal{BC}} = \mathcal{A}$.

We introduce a syntactic restriction on the type of expressions used to derive boxes. We will denote by Exp_0 those box expressions E for which there is no subexpression $F \,\Box\, G$ and a communication action a such that $a \in \mathsf{ex}_{\mathsf{box}(F)}$ and $\widehat{a} \in \mathsf{ex}_{\mathsf{box}(G)}$. By Box_0 we will denote boxes which can be derived from the box expressions in Exp_0.

Proposition 5.4 Let $\mathcal{B}$ be a box in Box_0.

1. There are no transitions t and u in $\mathcal{B}$ with conjugate labels and satisfying ${}^\bullet t = {}^\bullet u$ and $t^\bullet = u^\bullet$.
2. There is no $\imath$-labelled transition in $\mathcal{B}$ such that all the arcs adjacent to it have weight 2.

Proof. (1) By straightforward induction on the structure of the expression generating boxes in Box_0. (2) Follows immediately from the first part. $\square$

We then obtain a partial characterisation of the maximal synchronisation sets of boxes involving choice composition.

Proposition 5.5 Let $\mathcal{B} = (\mathcal{B}_1 \square \mathcal{B}_2)\,\mathsf{sy}\,A$ be a box in Box_0 and A be a synchronisation set such that

$$\mathsf{ex}_{\mathcal{B}} \cap (A \cup \{\imath\}) \quad\subseteq\quad \mathsf{ex}_{\mathcal{B}_1\,\mathsf{sy}\,A} \cap \mathsf{ex}_{\mathcal{B}_2\,\mathsf{sy}\,A}. \tag{3}$$

Then

$$\mathsf{max}_{\mathcal{B}} \quad=\quad \mathsf{covnoex}^{A}_{\mathcal{B}_1\mathcal{B}_2} \cap \mathsf{covint}^{A}_{\mathcal{B}_1\mathcal{B}_2} \cap \bigcap_{i=1}^{2} \mathsf{max}_{\mathcal{B}_i\,\mathsf{sy}\,A}.$$

Proof. Using notation from section 3.1.1, let $E = {}^\circ\mathcal{N} = {}^\circ\mathcal{M}$ and $X = \mathcal{N}^\circ = \mathcal{M}^\circ$, where

$$\mathcal{N} \;=\; \mathcal{B}_1 : \{({}^\circ\mathcal{B}_1, {}^\circ\mathcal{B}_2, \mathsf{e}), (\mathcal{B}_1{}^\circ, \mathcal{B}_2{}^\circ, \mathsf{x})\} \quad\text{and}\quad \mathcal{M} \;=\; \mathcal{B}_2 : \{({}^\circ\mathcal{B}_1, {}^\circ\mathcal{B}_2, \mathsf{e}), (\mathcal{B}_1{}^\circ, \mathcal{B}_2{}^\circ, \mathsf{x})\}.$$

Note that $\mathcal{B}_1 \square \mathcal{B}_2 = \mathcal{N} \cup \mathcal{M}$.

Suppose that $a \in \mathsf{max}_{\mathcal{B}}$ and $a \notin A$ (if $a \in A$ then a clearly belongs to the rhs of the equality from the formulation of this proposition). Using proposition 5.1(1,2) and the definition of the choice operator, one can see that A1–A5 are satisfied. Moreover, A6 holds which follows from (3) and $T_{EX} \cap T_{\mathcal{D}} = \mathsf{EX}_{\mathcal{D}}$, for $\mathcal{D} \in \{\mathcal{B}, \mathcal{B}_1\,\mathsf{sy}\,A, \mathcal{B}_2\,\mathsf{sy}\,A\}$ (note that the latter follows from propositions 5.1(1) and 5.4(2)). Hence, by proposition 3.10, $a \in \mathsf{max}_{\mathcal{N}\,\mathsf{sy}\,A}$. Since $\mathcal{N} \simeq \mathcal{B}_1$, we have $a \in \mathsf{max}_{\mathcal{B}_1\,\mathsf{sy}\,A}$ and, by symmetry, $a \in \mathsf{max}_{\mathcal{B}_2\,\mathsf{sy}\,A}$.

To show $a \in \mathsf{covint}^{A}_{\mathcal{N}\mathcal{M}} = \mathsf{covint}^{A}_{\mathcal{B}_1\mathcal{B}_2}$ we need to prove that if $(t,v) \in ((T_{\mathcal{N}} - \mathsf{EX}_{\mathcal{N}}) \times (T_{\mathcal{M}} - \mathsf{EX}_{\mathcal{M}})) \cap \mathsf{syn}_{\mathsf{a}}$ then there are $(u,w) \in (T_{\mathcal{N}} \times T_{\mathcal{M}}) \cap \mathsf{syn}_A$ such that $t \bowtie_{\mathcal{N}} u$ and $v \bowtie_{\mathcal{M}} w$. This, however, follows directly from proposition 3.11 (note that $t, v \notin T_{EX}$).

To show $a \in \mathsf{covnoex}^{A}_{\mathcal{N}\mathcal{M}} = \mathsf{covnoex}^{A}_{\mathcal{B}_1\mathcal{B}_2}$, by symmetry, we only need to prove that if $t \in \mathsf{EX}_{\mathcal{N}} = \mathsf{EX}_{\mathcal{B}_1}$ is an a-labelled transition, and $v \in U \in \mathsf{ccnoex}_{\mathcal{M}\,\mathsf{sy}\,A} = \mathsf{ccnoex}_{\mathcal{B}_2\,\mathsf{sy}\,A}$ is an $\widehat{a}$-labelled transition then there are $(u,w) \in (T_{\mathcal{N}} \times T_{\mathcal{M}}) \cap \mathsf{syn}_A$ such that $t \bowtie_{\mathcal{N}} u$ and $v \bowtie_{\mathcal{M}} w$. This, however, follows directly from proposition 3.11 (the proposition can be applied since $v \notin T_{EX}$, and if $t \in T_{EX}$ and $t \odot v \simeq_{\mathcal{B}\,\mathsf{sy}\,(A \cup \mathsf{a})} x$ where $x \in T_{\mathcal{M}\,\mathsf{sy}\,A}$ then $v\delta \bowtie_{\mathcal{M}\,\mathsf{sy}\,A} x$, contradicting the choice of v).

We have shown the ($\subseteq$) inclusion. To show the reverse one, we use corollary 3.9. That it can be applied is shown by the following argument. Suppose that $(t,v) \in (T_{\mathcal{N}} \times T_{\mathcal{M}}) \cap \mathsf{syn}_{\mathsf{a}}$, where a belongs to the rhs of the equality from the formulation of this proposition. We consider three cases.

Case 1: $t \notin \mathsf{EX}_{\mathcal{N}}$ and $v \notin \mathsf{EX}_{\mathcal{M}}$. By $a \in \mathsf{covint}^{A}_{\mathcal{B}_1\mathcal{B}_2}$, there are $(u,w) \in (T_{\mathcal{N}} \times T_{\mathcal{M}}) \cap \mathsf{syn}_A$ such that $t \bowtie_{\mathcal{N}} u$ and $v \bowtie_{\mathcal{M}} w$. Hence $tv \bowtie_{\mathcal{N} \cup \mathcal{M}} uw$.

Case 2: $t \in \mathsf{EX}_{\mathcal{N}}$ and $v \notin \mathsf{EX}_{\mathcal{M}}$ (note the argument is symmetric for $t \notin \mathsf{EX}_{\mathcal{N}}$ and $v \in \mathsf{EX}_{\mathcal{M}}$). If it is not the case that there is $z \in T_{\mathcal{M}\,\mathsf{sy}\,A}$ such that $tv \simeq_{\mathcal{B}} z$ then there is $U \in \mathsf{ccnoex}_{\mathcal{M}\,\mathsf{sy}\,A}$ such that $t \in U$ and we can use the fact that $a \in \mathsf{covnoex}^{A}_{\mathcal{B}_1\mathcal{B}_2}$ to reach the desired conclusion, i.e. to find u and w as in Case 1.

Case 3: $t \in \mathsf{EX}_{\mathcal{N}}$ and $v \in \mathsf{EX}_{\mathcal{M}}$. This produces a contradiction with proposition 5.4. $\square$

We finally observe that the various sets of communication actions introduced and used in this section are preserved through duplication equivalence and synchronisation.

Proposition 5.6 Let A, B, C be synchronisation sets and $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be boxes such that $\mathcal{B} \simeq \mathcal{D}$ and $\mathcal{C} \simeq \mathcal{E}$.

1. $\mathsf{set}_{\mathcal{B}} = \mathsf{set}_{\mathcal{D}}$ for $\mathsf{set} \in \{\mathsf{ex}, \mathsf{ccall}, \mathsf{ccint}, \mathsf{ccnoex}\}$.
2. $\mathsf{set}^{A}_{\mathcal{BC}} = \mathsf{set}^{A}_{\mathcal{DE}}$ for $\mathsf{set} \in \{\mathsf{covall}, \mathsf{covmix}, \mathsf{covint}, \mathsf{covnoex}\}$.
3. $\mathsf{set}^{A}_{\mathcal{BC}} = \mathsf{set}^{A}_{\mathcal{B}\,\mathsf{sy}\,B, \mathcal{C}\,\mathsf{sy}\,C}$ for $\mathsf{set} \in \{\mathsf{covall}, \mathsf{covmix}, \mathsf{covint}\}$. $\square$

Note that $\mathsf{covnoex}^{A}_{\mathcal{BC}} = \mathsf{covnoex}^{A}_{\mathcal{B}\,\mathsf{sy}\,B, \mathcal{C}\,\mathsf{sy}\,C}$ does not, in general, hold. For example, if $\mathcal{B} = \mathsf{box}(\widehat{a})$ and $\mathcal{C} = \mathsf{box}(\widehat{a} \,\square\, (a; \imath))$ then $\mathsf{covnoex}^{\emptyset}_{\mathcal{B}, \mathcal{C}\,\mathsf{sy}\,\mathsf{a}} = \mathcal{A}$ but $a \notin \mathsf{covnoex}^{\emptyset}_{\mathcal{BC}}$.

6 Box expressions

We now transfer the notion of duplication equivalence formulated for boxes to the domain of box expressions. Two box expressions, E and F, are *duplication equivalent* if $\mathsf{box}(E) \simeq \mathsf{box}(F)$. We denote this by $E \simeq F$. The maximal synchronisation set of a box expression E is defined as $\mathsf{max}_E = \mathsf{max}_{\mathsf{box}(E)}$. Clearly, many properties of duplication equivalence that hold for boxes can be transferred to box expressions. In particular, we immediately obtain that $\simeq$ is a congruence in the domain of box expressions.
Moreover, directly from propositions 3.4 and 3.5 we have the following.

Proposition 6.1 Let E and F be box expressions, and A and B be synchronisation sets.

1. $E\,\mathsf{sy}\,A\,\mathsf{sy}\,B \simeq E\,\mathsf{sy}\,(A \cup B)$.
2. If $E \simeq E\,\mathsf{sy}\,B$ and $A \subseteq B$ then $E \simeq E\,\mathsf{sy}\,A$.
3. If $E \simeq E\,\mathsf{sy}\,A$ and $E \simeq E\,\mathsf{sy}\,B$ then $E \simeq E\,\mathsf{sy}\,(A \cup B)$.
4. If $E \simeq F\,\mathsf{sy}\,A$ then $E \simeq E\,\mathsf{sy}\,A$. $\square$

The main aim of this paper is to axiomatise duplication equivalence of box expressions. When we approached this problem, it soon turned out that a crucial difficulty which had to be solved was the development of a structural characterisation of maximal synchronisation sets, both in order to obtain a set of sound axioms and to define normal form box expressions needed for a completeness proof. Such a characterisation is based on that obtained for boxes; we therefore define the expression counterparts of *ex*–transitions and choice context transitions as well as other notations introduced in the previous section. For a box expression E, let ex_E and potex_E be sets of actions defined by induction on the structure of E, as follows:

$$
\begin{array}{lcl@{\qquad}lcl}
\mathsf{ex}_{\beta} & = & \{\beta\} & \mathsf{potex}_{E \| F} & = & (\mathsf{ex}_E \cap \widehat{\mathsf{ex}_F}) \cup (\mathsf{ex}_F \cap \widehat{\mathsf{ex}_E}) \\
\mathsf{ex}_{E \square F} & = & \mathsf{ex}_E \cup \mathsf{ex}_F & \mathsf{potex}_{E \square F} & = & \mathsf{potex}_E \cup \mathsf{potex}_F \\
 & & & \mathsf{potex}_{E\,\mathsf{sy}\,A} & = & \mathsf{potex}_E
\end{array}
$$

$$
\mathsf{ex}_{E\,\mathsf{sy}\,A} = \begin{cases} \mathsf{ex}_E \cup \{\imath\} & \text{if } \mathsf{potex}_E \cap A \neq \emptyset \\ \mathsf{ex}_E & \text{otherwise} \end{cases}
$$

In all the remaining cases, ex_E and potex_E are defined as empty. The meaning of ex_E is that of $\mathsf{ex}_{\mathsf{box}(E)}$. The auxiliary set potex_E represents *potential* $\imath$-labelled *ex*-transitions which can be generated by applying synchronisation using the actions in potex_E. For example, $\mathsf{potex}_{(a \square b) \| (\widehat{a} \square \widehat{b})} = \mathsf{a} \cup \mathsf{b}$.

Proposition 6.2 For every box expression E, $\mathsf{ex}_E = \mathsf{ex}_{\mathsf{box}(E)}$ and $\mathsf{potex}_E = \lambda_{\mathsf{box}(E)}(T_0)$, where T_0 is the set of all transitions t, u in $\mathsf{box}(E)$ labelled with conjugate communication actions such that $tu \bowtie_{\mathsf{box}(E)} \delta$.

Proof. By induction on the structure of E. □

We next turn to choice context transitions. For a box expression E, let ccint_E and ccall_E be two sets of sets of communication actions defined by induction on the structure of E, as follows:

$$\begin{array}{lcllcl}
\mathsf{ccint}_\beta & = & \emptyset & \mathsf{ccint}_{[E*F*G]} & = & \mathsf{ccall}_E \cup \mathsf{ccall}_F \cup \mathsf{ccall}_G \\
\mathsf{ccint}_{E \,\square\, F} & = & \mathsf{ccint}_E \cup \mathsf{ccint}_F & \mathsf{ccint}_{E\|F} & = & \mathsf{ccall}_E \cup \mathsf{ccall}_F \\
\mathsf{ccint}_{E;F} & = & \mathsf{ccall}_E \cup \mathsf{ccall}_F & \mathsf{ccint}_{E\,\mathsf{sy}\,A} & = & \mathsf{ccint}_E
\end{array}$$

$$\mathsf{ccall}_E = \begin{cases} \mathsf{ccint}_E \cup \{\mathsf{ex}_E \cap \mathcal{A}\} & \text{if } \mathsf{ex}_E \cap \mathcal{A} \neq \emptyset \\ \mathsf{ccint}_E & \text{otherwise.} \end{cases}$$

Moreover, we define a set of sets of communication actions ccnoex_E, as follows:

$$\begin{array}{rlcl}
 & \mathsf{ccnoex}_{F \,\square\, G} & = & \mathsf{ccnoex}_F \cup \mathsf{ccnoex}_G \\
\text{and} & \mathsf{ccnoex}_{F\,\mathsf{sy}\,A} & = & \{C \in \mathsf{ccnoex}_F \mid \widehat{C} \cap \mathsf{ex}_F \cap A = \emptyset\}
\end{array}$$

and by setting $\mathsf{ccnoex}_E = \mathsf{ccint}_E$ in all the remaining cases.

Proposition 6.3 For every box expression E, $\mathsf{ccall}_E = \mathsf{ccall}_{\mathsf{box}(E)}$, $\mathsf{ccint}_E = \mathsf{ccint}_{\mathsf{box}(E)}$ and $\mathsf{ccnoex}_E = \mathsf{ccnoex}_{\mathsf{box}(E)}$.

Proof. By induction on the structure of E and using proposition 6.2. □

Let E, F and G be box expressions and A be a synchronisation set. We define:

$$\begin{array}{lcll}
\mathsf{covall}^A_{EF} & = & \mathsf{cov}^A(\mathsf{ccall}_E, \mathsf{ccall}_F) & \\
\mathsf{covnoex}^A_{EF} & = & \mathsf{cov}^A(\mathsf{ccnoex}_{E\,\mathsf{sy}\,A}, \{\mathsf{ex}_F\}) & \cap \;\; \mathsf{cov}^A(\mathsf{ccnoex}_{F\,\mathsf{sy}\,A}, \{\mathsf{ex}_E\}) \\
\mathsf{covmix}^A_{EF} & = & \mathsf{cov}^A(\mathsf{ccint}_E, \mathsf{ccall}_F) & \cap \;\; \mathsf{cov}^A(\mathsf{ccint}_F, \mathsf{ccall}_E) \\
\mathsf{covint}^A_{EF} & = & \mathsf{cov}^A(\mathsf{ccint}_E, \mathsf{ccint}_F) &
\end{array}$$

and $\mathsf{covall}^A_{EFG} = \bigcap\{\mathsf{covall}^A_{\mathcal{X}\mathcal{Y}} \mid \mathcal{X}, \mathcal{Y} \in \{E, F, G\}\}$. The sets of communication actions we have just defined are direct counterparts of similar notions introduced for boxes.

Proposition 6.4 Let E, F and G be expressions and A be a synchronisation set.

1. $\mathsf{set}^A_{EF} = \mathsf{set}^A_{\mathsf{box}(E)\mathsf{box}(F)}$, for $\mathsf{set} \in \{\mathsf{covall}, \mathsf{covint}, \mathsf{covmix}, \mathsf{covnoex}\}$.

2. $\mathsf{covall}^A_{EFG} = \mathsf{covall}^A_{\mathsf{box}(E)\mathsf{box}(F)\mathsf{box}(G)}$.

Proof. Follows from propositions 6.2 and 6.3. □

As a result, we can capture the relationship between max_E and the structure of E by adapting the results obtained for boxes. Below, for a pair of expressions E and F, simex_{EF} denotes the set of all synchronisation sets A such that

$$\mathsf{ex}_{(E \,\square\, F)\,\mathsf{sy}\,A} \cap (A \cup \{\imath\}) \;\; \subseteq \;\; \mathsf{ex}_{E\,\mathsf{sy}\,A} \cap \mathsf{ex}_{F\,\mathsf{sy}\,A}.$$

Proposition 6.5 Let E_i $(i = 1, 2, 3)$ be box expressions and A be a synchronisation set. Then

$$\begin{array}{rcll} \mathsf{max}_{(E_1;E_2)\,\mathsf{sy}\,A} & = & \mathsf{covall}^A_{E_1E_2} & \cap \bigcap_{i=1}^{2} \mathsf{max}_{E_i\,\mathsf{sy}\,A} \\ \mathsf{max}_{(E_1\|E_2)\,\mathsf{sy}\,A} & = & \mathsf{covall}^A_{E_1E_2} & \cap \bigcap_{i=1}^{2} \mathsf{max}_{E_i\,\mathsf{sy}\,A} \\ \text{and}\quad \mathsf{max}_{[E_1*E_2*E_3]\,\mathsf{sy}\,A} & = & \mathsf{covall}^A_{E_1E_2E_3} & \cap \bigcap_{i=1}^{3} \mathsf{max}_{E_i\,\mathsf{sy}\,A}. \end{array}$$

Moreover, if $(E_1 \square E_2)\,\mathsf{sy}\,A \in \mathsf{Exp}_0$ and $A \in \mathsf{simex}_{E_1E_2}$ then

$$\mathsf{max}_{(E_1\square E_2)\,\mathsf{sy}\,A} = \mathsf{covnoex}^A_{E_1E_2} \cap \mathsf{covint}^A_{E_1E_2} \cap \bigcap_{i=1}^{2} \mathsf{max}_{E_i\,\mathsf{sy}\,A}.$$

Proof. Follows from propositions 5.3, 5.5 and 6.4. $\square$

The above characterisation of maximal synchronisation sets of expressions involving the choice operator is only partial. For that reason, we will now provide two results on specific usages of choice composition.

Proposition 6.6 Let $E = (E_1\|E_2)\,\mathsf{sy}\,A$ be a box expression in Exp_0 and a be a communication action such that $a \in \mathsf{max}_{E\square\imath} - \mathsf{max}_E$. Then $\mathsf{ex}_{E_1} \cap \widehat{\mathsf{ex}_{E_2}} \cap \mathsf{a} \neq \emptyset$ and

$$a \in \mathsf{covmix}^A_{E_1E_2} \cap \bigcap_{i=1}^{2} \mathsf{max}_{E_i\,\mathsf{sy}\,A}.$$

Proof. Let $\mathcal{B} = \mathsf{box}(\imath)$ and $\mathcal{B}_i = \mathsf{box}(E_i)$, for $i = 1, 2$. Since $|{}^\circ\mathcal{B}| = |\mathcal{B}^\circ| = 1$, we may assume that $(\mathcal{B}_1\|\mathcal{B}_2) \square \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ where

$$\mathcal{B}_3 = \mathcal{B} \oplus \{(s, {}^\circ\mathcal{B}_1 \cup {}^\circ\mathcal{B}_2, \mathsf{e}), (r, \mathcal{B}_1{}^\circ \cup \mathcal{B}_2{}^\circ, \mathsf{x})\}$$

and s and r are places satisfying $\{s\} = {}^\circ\mathcal{B}$ and $\{r\} = \mathcal{B}^\circ$. We have

$$\begin{array}{rcl} (\mathcal{B}_1 \cup \mathcal{B}_2)\,\mathsf{sy}\,A & \not\simeq & (\mathcal{B}_1 \cup \mathcal{B}_2)\,\mathsf{sy}\,(A \cup \mathsf{a}) \\ \text{and}\quad (\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)\,\mathsf{sy}\,A & \simeq & (\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)\,\mathsf{sy}\,(A \cup \mathsf{a}) \end{array}$$

Since $(\mathcal{B}_1 \cup \mathcal{B}_2)\,\mathsf{sy}\,A$ and $(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)\,\mathsf{sy}\,A$ differ only by a single $\imath$–labelled transition $t \in \mathsf{EX}_{\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3}$, from proposition 2.4(2) it follows that there are must be a–synchronisable transitions u, w in $\mathcal{B}_1 \cup \mathcal{B}_2$ such that $\delta \bowtie_{\mathcal{B}_1\cup\mathcal{B}_2} uw$. Hence $\mathsf{ex}_{E_1} \cap \widehat{\mathsf{ex}_{E_2}} \cap \mathsf{a} \neq \emptyset$.

To prove the second part, we again refer to the results and notation from section 3.1.1, and define: $\mathcal{N} = \mathcal{B}_1$, $\mathcal{M} = \mathcal{B}_2 \cup \mathcal{B}_3$, $E = {}^\circ\mathcal{N}$ and $X = \mathcal{N}^\circ$. It is easy to see, using proposition 5.1, that A1-A5 hold. Moreover, the only EX-transitions (if any) in $(\mathcal{N} \cup \mathcal{M})\,\mathsf{sy}\,A$ are those in $\mathcal{N}\,\mathsf{sy}\,A$. Hence A6 also holds and, by proposition 3.10, we obtain that $a \in \mathsf{max}_{E_1\,\mathsf{sy}\,A}$ and, by symmetry, that $a \in \mathsf{max}_{E_2\,\mathsf{sy}\,A}$. We can also observe that proposition 3.11 can be applied for $(t, v) \in ((T_\mathcal{N} - \mathsf{EX}_\mathcal{N}) \times T_\mathcal{M}) \cap \mathsf{syn}_\mathsf{a}$ since then $t, v \notin T_{EX}$. From this, and by symmetry, $a \in \mathsf{covmix}^A_{E_1E_2}$. $\square$

Proposition 6.7 Let $a \in \mathcal{A}$ be such that $a \in \mathsf{max}_{(E\square\beta_1\square\ldots\square\beta_k)\,\mathsf{sy}\,A}$ where $\beta_1, \ldots \beta_k$ are actions and $E \square \beta_1 \square \ldots \square \beta_k$ is an expression in Exp_0.

1. If the topmost operator of E is sequence or iteration then $a \in \mathsf{max}_E$.
2. If the topmost operator of E is parallel composition and $a \notin \mathsf{max}_E$ then $a \in \mathsf{max}_{(E\,\mathsf{sy}\,A\square)\imath}$ and $\beta_i = \imath$, for some $i \leq k$.

Proof. Let $\mathcal{B} = \mathsf{box}(E)$ and $\mathcal{B}_i = \mathsf{box}(\beta_i)$ for $i = 1, \ldots, k$. Since $|{}^\circ\mathcal{B}_i| = |\mathcal{B}_i{}^\circ| = 1$ for every $i \leq k$, we may assume that

$$\mathcal{N} = \mathsf{box}(E \,\square\, \beta_1 \,\square\, \ldots \,\square\, \beta_k) = \mathcal{B} \cup \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_k$$

where in the above, for $i = 1, \ldots, k$,

$$\mathcal{D}_i = \mathcal{B}_i \oplus \{(s_i, {}^\circ\mathcal{B}, \mathsf{e}), (r_i, \mathcal{B}^\circ, \mathsf{x})\}$$

and s_i and r_i are places satisfying $\{s_i\} = {}^\circ\mathcal{B}_i$ and $\{r_i\} = \mathcal{B}_i{}^\circ$.

By proposition 3.5(3), $\mathcal{N}\,\mathsf{sy}\,A \cong \mathcal{N}\,\mathsf{sy}\,(A \cup \mathsf{a})$. If $a \notin \mathsf{max}_E$ then, by proposition 2.4(1,2) and $a \in \mathsf{max}_{(E \square \beta_1 \square \ldots \square \beta_k)\,\mathsf{sy}\,A}$, there are $t, u \in T_\mathcal{B}$ such that $(t, u) \in \mathsf{syn}_\mathsf{a}$ and one of the following holds:

(a) There is $w \in T_{\mathcal{D}_1 \cup \ldots \cup \mathcal{D}_k}$ such that $t \odot u \simeq_{\mathcal{N}\,\mathsf{sy}\,(A \cup \mathsf{a})} w$.

(b) There are $v \in T_\mathcal{B}$ and $w \in T_{\mathcal{D}_1 \cup \ldots \cup \mathcal{D}_k}$ such that $t \odot u \simeq_{\mathcal{N}\,\mathsf{sy}\,(A \cup \mathsf{a})} v \odot w$.

(c) There are $v, w \in T_{\mathcal{D}_1 \cup \ldots \cup \mathcal{D}_k}$ such that $t \odot u \simeq_{\mathcal{N}\,\mathsf{sy}\,(A \cup \mathsf{a})} v \odot w$.

Clearly, (c) is impossible because the weights of all the arcs adjacent to $v \odot w$ would be equal to 2, contradicting proposition 5.4(2). We then consider three cases.

Case 1: $E = F; G$. Then (a) leads to a contradiction since $t \odot u$ is connected to at least one internal place in $\mathcal{N}\,\mathsf{sy}\,(A \cup \mathsf{a})$. That (b) also leads to a contradiction can be shown in the following way. Since ${}^\bullet w = {}^\circ\mathcal{B}$ and $w^\bullet = \mathcal{B}^\circ$ it must be the case that, without loss of generality, t is a transition from $\mathsf{box}(F)$ and u is a transition from $\mathsf{box}(G)$. Then, again without loss of generality, v is a transition from $\mathsf{box}(F)$. Note now that $\mathsf{box}(E)$ is the union of $\mathcal{F}' = \mathsf{box}(F) : \{(\mathsf{box}(F)^\circ, {}^\circ\mathsf{box}(G), \mathsf{i})\}$ and $\mathcal{G}' = \mathsf{box}(G) : \{(\mathsf{box}(F)^\circ, {}^\circ\mathsf{box}(G), \mathsf{i})\}$ and observe that there is a non-entry place s in $\mathsf{box}(E)$ which is also in ${}^\bullet u$. Hence, s also has to be in ${}^\bullet v$. But the only places shared by $\mathcal{F}'$ and $\mathcal{G}'$ are those in $\mathsf{box}(F)^\circ \otimes {}^\circ\mathsf{box}(G)$. This means that $\mathsf{box}(F)^\circ \cap {}^\bullet v \neq \emptyset$, a contradiction.

Case 2: $E = [F * G * H]$. We proceed similarly as in Case 1.

Case 3: $E = F \| G$. Then (b) leads to a contradiction since we obtain from proposition 5.1(2) that, without loss of generality, ${}^\bullet t \subseteq {}^\circ\mathsf{box}(F)$, ${}^\bullet u \subseteq {}^\circ\mathsf{box}(G)$, $t^\bullet \subseteq \mathsf{box}(F)^\circ$ and $u^\bullet \subseteq \mathsf{box}(G)^\circ$. Moreover, ${}^\circ\mathcal{B} \subseteq {}^\bullet t \cup {}^\bullet u$ and $\mathcal{B}^\circ \subseteq t^\bullet \cup u^\bullet$. Hence $t \in \mathsf{EX}_{\mathsf{box}(F)}$ and $u \in \mathsf{EX}_{\mathsf{box}(G)}$. But this means that ${}^\bullet v = v^\bullet = \emptyset$. If (a) holds then, clearly, $\beta_i = \imath$, where i is such that w is the only transition of $\mathcal{B}_i$. Moreover, $a \in \mathsf{max}_{E\,\mathsf{sy}\,A \square \imath}$ can be shown easily using proposition 2.4(2). $\square$

We end this section with a useful result which states that the various sets and relations introduced in this section are preserved through duplication equivalence and synchronisation.

Proposition 6.8 Let A, B, C be synchronisation sets and E, F, G, H be boxes such that $E \simeq G$ and $F \simeq H$.

1. $\mathsf{set}_E = \mathsf{set}_G$ for $\mathsf{set} \in \{\mathsf{ex}, \mathsf{ccall}, \mathsf{ccint}, \mathsf{ccnoex}\}$.
2. $\mathsf{set}_E = \mathsf{set}_{E\,\mathsf{sy}\,A}$ for $\mathsf{set} \in \{\mathsf{ccall}, \mathsf{ccint}\}$.
3. $\mathsf{set}^A_{EF} = \mathsf{set}^A_{GH}$ for $\mathsf{set} \in \{\mathsf{covall}, \mathsf{covmix}, \mathsf{covint}, \mathsf{covnoex}\}$.
4. $\mathsf{set}^A_{EF} = \mathsf{set}^A_{E\,\mathsf{sy}\,B, F\,\mathsf{sy}\,C}$ for $\mathsf{set} \in \{\mathsf{covall}, \mathsf{covmix}, \mathsf{covint}\}$.
5. $\mathsf{simex}_{EF} = \mathsf{simex}_{GH}$.

Proof. Follows from propositions 5.6, 6.2, 6.3 and 6.4. $\square$

7 An axiomatisation of duplication equivalence

The axioms for duplication equivalence of box expressions are structured into six groups. Below, β stands for an arbitrary action, A and B for synchronisation sets, and a for a communication action.

Structural Identities. The first group of axioms (STR1-STR5) capture some basic structural identities. The axioms are sound not only with respect to duplication equivalence, but also with respect to net isomorphism. What they express is that the choice, parallel and sequential compositions are associative[6], and that the first two are also commutative operators.

$$
\begin{array}{rcll}
(E;F);G &=& E;(F;G) & \mathsf{STR1} \\
(E \square F) \square G &=& E \square (F \square G) & \mathsf{STR2} \\
E \square F &=& F \square E & \mathsf{STR3} \\
(E\|F)\|G &=& E\|(F\|G) & \mathsf{STR4} \\
E\|F &=& F\|E & \mathsf{STR5} \\
\beta &=& \beta \,\mathsf{sy}\, A & \mathsf{PROP1} \\
E &=& E \,\mathsf{sy}\, \emptyset & \mathsf{PROP2} \\
E \,\mathsf{sy}\, A \,\mathsf{sy}\, B &=& E \,\mathsf{sy}\, (A \cup B) & \mathsf{PROP3} \\
(E;F) \,\mathsf{sy}\, A &=& ((E \,\mathsf{sy}\, A);(F \,\mathsf{sy}\, A)) \,\mathsf{sy}\, A & \mathsf{PROP4} \\
(E \square F) \,\mathsf{sy}\, A &=& ((E \,\mathsf{sy}\, A) \square (F \,\mathsf{sy}\, A)) \,\mathsf{sy}\, A & \mathsf{PROP5} \\
(E\|F) \,\mathsf{sy}\, A &=& ((E \,\mathsf{sy}\, A)\|(F \,\mathsf{sy}\, A)) \,\mathsf{sy}\, A & \mathsf{PROP6} \\
[E * F * G] \,\mathsf{sy}\, A &=& [(E \,\mathsf{sy}\, A) * (F \,\mathsf{sy}\, A) * (G \,\mathsf{sy}\, A)] \,\mathsf{sy}\, A & \mathsf{PROP7} \\
\beta \square \beta &=& \beta & \mathsf{DUPL}
\end{array}
$$

Propagation of synchronisation. The first two of the next group of axioms (PROP1-PROP7) express simple (structural) facts about synchronisation, namely that applying synchronisation to a single action expression, or using the empty synchronisation set, has no effect at all. The third axiom allows one to collapse consecutive applications of the synchronisation operator. The remaining four axioms amount to saying that synchronisation propagates through the four composition operators.

Duplication. This group comprises only one axiom (DUPL). It captures the essence of duplication equivalence whereby a choice between two copies of the same action is ignored.

***ex*-actions.** The next axiom (EX) is used to deal with *ex*–actions as it allows these to be moved within a box expression. This is necessary, in particular, in order to make an expression with the main choice composition connective satisfy the first of the premises in the next axiom (LIFT1).

Lifting of synchronisation. The following four axioms (LIFT1-LIFT4) allow one to lift synchronisation sets to a higher level in the syntax tree of a box expression. The main application of these axioms is in the construction of maximal synchronisation sets.

Internal actions. The last axioms (INT1-INT2) capture two different ways in which a syntactically generated internal action can find its duplicate generated through synchronisation.

From now on we restrict ourselves to the box expressions which belong to Exp_0. Note that by applying any of the axioms to a box expression in Exp_0 one always produces an expression which also belongs to Exp_0. We will also assume that the set of communication actions $\mathcal{A}$ is finite.

[6]We can therefore omit the parentheses in nested applications of sequence, choice and parallel composition operators.

$$\frac{\beta \in A \;\Rightarrow\; \forall B \in \mathsf{ccnoex}_{E\,\mathsf{sy}\,A} : \widehat{\beta} \notin B}{(E\,\mathsf{sy}\;A) \Box \beta = (E \Box \beta)\;\mathsf{sy}\;A} \qquad \mathsf{EX}$$

$$\frac{A \in \mathsf{simex}_{E\,\mathsf{sy\,a},F\,\mathsf{sy\,a}} \text{ and } a \in \mathsf{covint}^{A}_{EF} \cap \mathsf{covnoex}^{A}_{E\,\mathsf{sy\,a},F\,\mathsf{sy\,a}}}{((E\,\mathsf{sy\,a}) \Box (F\,\mathsf{sy\,a}))\;\mathsf{sy}\;A = (E \Box F)\;\mathsf{sy}\;A\;\mathsf{sy\,a}} \qquad \mathsf{LIFT1}$$

$$\frac{a \in \mathsf{covall}^{A}_{EF}}{((E\,\mathsf{sy\,a}); (F\,\mathsf{sy\,a}))\;\mathsf{sy}\;A = (E;F)\;\mathsf{sy}\;A\;\mathsf{sy\,a}} \qquad \mathsf{LIFT2}$$

$$\frac{a \in \mathsf{covall}^{A}_{EF}}{((E\,\mathsf{sy\,a})\|(F\,\mathsf{sy\,a}))\;\mathsf{sy}\;A = (E\|F)\;\mathsf{sy}\;A\;\mathsf{sy\,a}} \qquad \mathsf{LIFT3}$$

$$\frac{a \in \mathsf{covall}^{A}_{EFG}}{[(E\,\mathsf{sy\,a}) * (F\,\mathsf{sy\,a}) * (G\,\mathsf{sy\,a})]\;\mathsf{sy}\;A = [E * F * G]\;\mathsf{sy}\;A\;\mathsf{sy\,a}} \qquad \mathsf{LIFT4}$$

$$(((E \Box a)\|(F \Box \widehat{a}))\;\mathsf{sy\,a}) \Box \imath = ((E \Box a)\|(F \Box \widehat{a}))\;\mathsf{sy\,a} \qquad \mathsf{INT1}$$

$$\frac{a \in \mathsf{covmix}^{A}_{EF} \cap \mathsf{ex}_{E} \cap \widehat{\mathsf{ex}_{F}}}{(((E\,\mathsf{sy\,a})\|(F\,\mathsf{sy\,a}))\;\mathsf{sy}\;A) \Box \imath = (E\|F)\;\mathsf{sy}\;A\;\mathsf{sy\,a}} \qquad \mathsf{INT2}$$

8 Soundness of the axiom system

The first property of the axiom system we establish is soundness. Using the results obtained for nets and, in particular, the structural characterisation of the maximal synchronisation sets, it is fairly routine to show that the axiom system is indeed sound. If two box expressions, E and F, can be shown to be equivalent using these axioms, we will write $E \equiv F$.

Theorem 8.1 For every box expression E in Exp_0, if $E \equiv F$ then $E \simeq F$.

Proof. As shown in [2], duplication equivalence is a congruence with respect to all the operators used in this paper. Hence it suffices to show that each of the axioms is sound. This is true for STR1-STR5 (since these are sound w.r.t. net isomorphism), PROP1, PROP2 and DUPL (which follows directly from the definition of $\square$, sy and $\simeq$), PROP3 (using proposition 6.1(1)), PROP4-PROP7 (using proposition 3.7 and the property that synchronisation distributes over place multiplication and place addition), LIFT1-LIFT4 (using propositions 6.8(4, for covint) and 6.5, as well as axioms PROP4-PROP7) and EX (using propositions 2.4(2) and 6.2), INT1 and INT2 (using proposition 2.4(2)). $\square$

Note that EX, DUPL and STR2 imply the soundness of two useful derived axioms, EX1 and EX2, where in the former $E \square \beta$ can be equal to β.

$$((E \square \beta)\ \mathsf{sy}\ A) \square \beta \;=\; (E \square \beta)\ \mathsf{sy}\ A \qquad \mathsf{EX1}$$

$$\frac{\beta \notin A}{(E\ \mathsf{sy}\ A) \square \beta = (E \square \beta)\ \mathsf{sy}\ A} \qquad \mathsf{EX2}$$

9 Completeness of the axiom system

The proof of completeness is much more involved and is structured into two parts. The first one deals with maximal synchronisation sets showing that it is always possible to make the maximal synchronisation set of a box expression the outermost synchronisation, i.e., for every box expression E in Exp_0, $E \equiv E\,\mathsf{sy}\,\mathsf{max}_E$. The proof of this result relies heavily on the structural characterisation of the maximal synchronisation sets of box expressions. The second part begins with the development of a normal form for box expressions and ends up with the completeness proof.

We first introduce an auxiliary notion of context. A *context* is a term $\aleph$ derived from the following syntax:

$$\aleph \;::=\; x\ \mathsf{sy}\ A \;\mid\; (\aleph \square \aleph)\ \mathsf{sy}\ A \;\mid\; (\aleph \| \aleph)\ \mathsf{sy}\ A \;\mid\; (\aleph ; \aleph)\ \mathsf{sy}\ A \;\mid\; [\aleph * \aleph * \aleph]\ \mathsf{sy}\ A$$

where A is a synchronisation set and x is a place-holder (variable). It is assumed that within a given context, all the place holders are distinct. An *n-context* is a context $\aleph$ with exactly n place holders. It can be denoted by $\aleph[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ are the place-holders of $\aleph$ listed in the order in which they occur in $\aleph$. For every $i \leq n$ we will denote by $A^i_\aleph$ the synchronisation set such that $\mathsf{sy}\ A^i_\aleph$ was the synchronisation directly applied to the i-th place-holder, i.e., x_i. Moreover, for all $i, j \leq n$ we will denote by $A^{ij}_\aleph$ the union of all sets A such that x_i and x_j are in the scope an application of $\mathsf{sy}\ A$. For example, if

$$\aleph[x, y, z] = ((((x\ \mathsf{sy}\ \mathsf{a}) \| (y\ \mathsf{sy}\ \mathsf{c}))\ \mathsf{sy}\ \mathsf{d}) \| (z\ \mathsf{sy}\ \mathsf{e}))\ \mathsf{sy}\ \mathsf{f}$$

then $A^2_\aleph = \mathsf{c}$, $A^{22}_\aleph = \mathsf{c} \cup \mathsf{d} \cup \mathsf{f}$ and $A^{12}_\aleph = \mathsf{d} \cup \mathsf{f}$. We will use $\aleph[p_1, \ldots, p_n]$ to denote the term resulting from replacing the place holders $x_1, \ldots, x_n$ by terms $p_1, \ldots, p_n$. Note that if each p_i is a box expression then so is $\aleph[p_1, \ldots, p_n]$. A context $\aleph$ is *saturated* (*reduced*) if, for every subcontext

$$(\ldots(\ldots)\,\mathsf{sy}\,A\ldots)\,\mathsf{sy}\,B$$

of $\aleph$, we have $B \subseteq A$ (resp. $B \cap A = \emptyset$).

A box expression E is in *standard form* if $E = \aleph[\beta_1, \ldots, \beta_n]$, for some n-context $\aleph$ and actions $\beta_1, \ldots, \beta_n$. Moreover, E is saturated (reduced) if $\aleph$ is saturated (resp. reduced). For a box expression E (or context $\aleph$), we use $\langle E \rangle$ (resp. $\langle \aleph \rangle$) to denote the term resulting from deleting all the occurrences of the synchronisation operator. Note that $\langle E \rangle$ is always a box expression. For example, $E = ((a\,\mathsf{sy}\,\mathsf{a})\|(b\,\mathsf{sy}\,(\mathsf{a} \cup \mathsf{c})))\,\mathsf{sy}\,\mathsf{a}$ is a saturated box expression in standard form such that $\langle E \rangle = a\|b$.

It is always possible to transform an expression into one in standard form, saturated or reduced. In the first of the two results that follow, a suitable F can be obtained by applying axioms PROP2 and PROP3, and in the second one by applying axioms PROP4-PROP7.

Proposition 9.1 For every expression E there is an expression in standard form F such that $\langle E \rangle = \langle F \rangle$ and $E \equiv F$. □

Proposition 9.2 Let $E = \aleph[E_1, \ldots, E_n]$ be a box expression in standard form. Then there is a saturated (resp. reduced) n-context $\aleph'$ such that $E \equiv \aleph'[E_1, \ldots, E_n]$ and $\langle \aleph \rangle = \langle \aleph' \rangle$ and $A^{ij}_\aleph = A^{ij}_{\aleph'}$, for all $i, j \leq n$. □

For example, if $E = ((a\,\mathsf{sy}\,\mathsf{a})\|(b\,\mathsf{sy}\,\mathsf{c}))\,\mathsf{sy}\,\mathsf{a}$ then $E \equiv ((a\,\mathsf{sy}\,\mathsf{a})\|(b\,\mathsf{sy}\,(\mathsf{a} \cup \mathsf{c})))\,\mathsf{sy}\,\mathsf{a}$ (exemplifying saturation) and $E \equiv ((a\,\mathsf{sy}\,\emptyset)\|(b\,\mathsf{sy}\,\mathsf{c}))\,\mathsf{sy}\,\mathsf{a}$ (exemplifying reduction).

9.1 Constructing maximal synchronisation sets

This section contains two results. The first, an auxiliary one, states that it is always possible to propagate an *ex*–action to the outside of a box expression.

Proposition 9.3 For every box expression E in Exp_0 and every $\beta \in \mathsf{ex}_E$, $E \equiv E \square \beta$.

Proof. By DUPL and STR2, it suffices to show that $E \equiv F \square \beta$, for some expression F. We proceed by induction on the structure of $\langle E \rangle$. By proposition 9.1, we may assume that E is in standard form. The base case is $E = \beta\,\mathsf{sy}\,A$. Then $\mathsf{ex}_E = \{\beta\}$ and, by EX1, $\beta\,\mathsf{sy}\,A \equiv (\beta\,\mathsf{sy}\,A) \square \beta$. The induction step is split into three cases.

Case 1: $E = (F; G)\,\mathsf{sy}\,A$ or $E = [F * G * H]\,\mathsf{sy}\,A$. Then $\mathsf{ex}_E = \emptyset$ and there is nothing to prove.

Case 2: $E = (F\|G)\,\mathsf{sy}\,A$. Then, by the definition of ex, $\beta = \imath$ and there is $a \in A$ such that $a \in \mathsf{ex}_F$ and $\widehat{a} \in \mathsf{ex}_G$. By the induction hypothesis, $F \equiv F \square a$ and $G \equiv G \square \widehat{a}$. Thus, by INT1, PROP3 and EX2,

$$E \equiv ((F \square a)\|(G \square \widehat{a}))\,\mathsf{sy}\,\mathsf{a}\,\mathsf{sy}\,A \equiv (((F \square a)\|(G \square \widehat{a}))\,\mathsf{sy}\,\mathsf{a} \square \imath)\,\mathsf{sy}\,A \equiv ((F \square a)\|(G \square \widehat{a}))\,\mathsf{sy}\,\mathsf{a}\,\mathsf{sy}\,A \square \imath.$$

Case 3: $E = (F \square G)\,\mathsf{sy}\,A$. Then, by proposition 9.2, there is a saturated n-context $\aleph$, $n \geq 2$, and box expressions $E_1, \ldots, E_n$ such that $E \equiv \aleph[E_1, \ldots, E_n]$, $\aleph$ uses only choice and synchronisation, and, for every $i \leq n$, the topmost operator of E_i (if E_i is not a single action expression) is neither choice nor synchronisation. We now observe that from $A^i_\aleph = A^{ii}_\aleph$ for every $i \leq n$ (as $\aleph$ is saturated) and the definition of ex, it follows that there is $k \leq n$ such that $\beta \in \mathsf{ex}_{E_k\,\mathsf{sy}\,A^k_\aleph}$. Hence, by the induction hypothesis, $E_k\,\mathsf{sy}\,A^k_\aleph \equiv (E_k\,\mathsf{sy}\,A^k_\aleph) \square \beta$. Then, by repeatedly applying STR2, STR3 and EX1 , one we can show that $E \equiv H \square \beta$, for some H. □

The next result shows that it is always possible to make the maximal synchronisation set of an expression the outermost synchronisation.

Proposition 9.4 For every expression E in Exp_0, $E \equiv E \,\mathsf{sy}\, \mathsf{max}_E$.

Proof. By $\mathsf{PROP3}$ and the finiteness of $\mathcal{A}$, it suffices to show that for every expression E and action $a \in \mathsf{max}_E$, $E \equiv F \,\mathsf{sy}\, \mathsf{a}$ for some expression F. We proceed by induction on the structure of $\langle E \rangle$. By proposition 9.1, we may assume that E is in standard form (note that applying proposition 9.1 does not change the expression if we disregard the occurrences of synchronisation operator; hence as far as the inductive proof on the structure of $\langle E \rangle$ is concerned, applying the proposition is harmless). The base case is $E = \beta \,\mathsf{sy}\, A$. Then, using $\mathsf{PROP1}$ and $\mathsf{PROP3}$, we obtain

$$\beta \,\mathsf{sy}\, A \equiv \beta \,\mathsf{sy}\, \mathcal{A} \,\mathsf{sy}\, A \equiv \beta \,\mathsf{sy}\, \mathcal{A} \equiv \beta \,\mathsf{sy}\, \mathcal{A} \,\mathsf{sy}\, \mathsf{a}.$$

In the induction step, if $E = (F;G) \,\mathsf{sy}\, A$ then, by proposition 6.5, $a \in \mathsf{max}_{F \,\mathsf{sy}\, A} \cap \mathsf{max}_{G \,\mathsf{sy}\, A}$ and $\mathsf{a} \in \mathsf{covall}^A_{FG}$. By the induction hypothesis, $F \equiv F \,\mathsf{sy}\, \mathsf{a}$ and $G \equiv G \,\mathsf{sy}\, \mathsf{a}$. Hence, by $\mathsf{LIFT2}$,

$$E \equiv ((F \,\mathsf{sy}\, \mathsf{a}); (G \,\mathsf{sy}\, \mathsf{a})) \,\mathsf{sy}\, A \equiv (F;G) \,\mathsf{sy}\, A \,\mathsf{sy}\, \mathsf{a}.$$

If $E = (F \| G) \,\mathsf{sy}\, A$ or $E = [E * F * G] \,\mathsf{sy}\, A$ then we proceed in a similar way.

The only complicated case to consider is that of $E = (F \Box G) \,\mathsf{sy}\, A$. Then there is an n-context $\aleph'$, $n \geq 2$, and box expressions $E_1, \ldots, E_n$ such that $E = \aleph'[E_1, \ldots, E_n]$, $\aleph'$ uses only choice and synchronisation, and, for every $i \leq n$, the topmost operator of every E_i (if E_i is not a single action expression) is different from choice and synchronisation.

For every $i \leq n$, let $F_i = E_i \Box a_1 \Box \ldots \Box a_m$ where $\{a_1, \ldots, a_m\} = \{E_j \mid E_j \in A^{ij}_{\aleph} \wedge j \neq i\}$ (if $m = 0$ then $F_i = E_i$). We now perform three-stage transformation of E into an expression whose structure will allow us to apply proposition 6.5. We can assume, by proposition 9.2, that E is reduced.

In the first stage, by applying $\mathsf{STR2}$, $\mathsf{STR3}$, DUPL and $\mathsf{EX2}$, we obtain that $E \equiv F$ where $F = \aleph'[F_1, \ldots, F_n]$. In the second stage, if $\imath \in \mathsf{ex}_E$ then we can, using proposition 9.3 and axioms $\mathsf{STR2}$, $\mathsf{STR3}$, DUPL and $\mathsf{EX2}$, transform F into an equivalent expression $H = \aleph'[H_1, \ldots, H_n]$ where $H_i = F_i \Box \imath$, for every $i \leq n$. In the third stage, using proposition 9.2, we can transform H into an equivalent expression $J = \aleph[H_1, \ldots, H_n]$ such that $\langle \aleph \rangle = \langle \aleph' \rangle$, $\aleph$ is saturated and $A^{ij}_{\aleph} = A^{ij}_{\aleph'}$, for all $i, j \leq n$.

Now we can observe that for every subcontext $(\aleph_1[x_k, \ldots, x_l] \Box \aleph_2[x_{l+1}, \ldots, x_m]) \,\mathsf{sy}\, A^{km}_{\aleph}$ of $\aleph$, the expression $(\aleph_1[H_k, \ldots, H_l] \Box \aleph_2[H_{l+1}, \ldots, H_m]) \,\mathsf{sy}\, A^{km}_{\aleph}$ does satisfy the pre-condition in the second part of proposition 6.5. In particular, this and the fact that $\aleph$ is saturated imply that, for every $i \leq n$, $a \in \mathsf{max}_{H_i \,\mathsf{sy}\, A^i_{\aleph}}$.

We now prove that for every $i \leq n$, $H_i \,\mathsf{sy}\, A^i_{\aleph} \equiv K \,\mathsf{sy}\, \mathsf{a}$, for some K. We first observe that if $a \in A^{ii}_{\aleph} = A^i_{\aleph}$ then we are done by $\mathsf{PROP3}$. Moreover, if it is the case that $H_i = E_i$ then we can use the induction hypothesis. So suppose that $a \notin A^i_{\aleph}$ and $H_i = (E_i \Box \psi)$. Note that in such a case, by construction, neither a nor $\widehat{a}$ occurs in ψ. We consider four cases:

Case 1: $E_i = \beta$. Then $H_i \equiv H_i \,\mathsf{sy}\, \mathsf{a}$ which follows from $\mathsf{PROP1}$, $\mathsf{STR2}$, $\mathsf{STR3}$, $\mathsf{EX2}$ and the fact that ψ does not contain a nor $\widehat{a}$. Hence, by $\mathsf{PROP3}$, $H_i \,\mathsf{sy}\, A^i_{\aleph} \equiv H_i \,\mathsf{sy}\, A^i_{\aleph} \,\mathsf{sy}\, \mathsf{a}$.

Case 2: $E_i = F;G$. Then, by proposition 6.7(1), $a \in \mathsf{max}_{E_i \,\mathsf{sy}\, A^i_{\aleph}}$ and, since $\langle E_i \rangle$ is a proper subexpression of $\langle E \rangle$, by the induction hypothesis and axioms $\mathsf{PROP3}$ and $\mathsf{PROP5}$, we obtain

$$(E_i \Box \psi) \,\mathsf{sy}\, A^i_{\aleph} \equiv ((E_i \,\mathsf{sy}\, A^i_{\aleph}) \Box (\psi \,\mathsf{sy}\, A^i_{\aleph})) \,\mathsf{sy}\, A^i_{\aleph} \equiv ((E_i \,\mathsf{sy}\, \mathsf{a} \,\mathsf{sy}\, A^i_{\aleph}) \Box (\psi \,\mathsf{sy}\, A^i_{\aleph})) \,\mathsf{sy}\, A^i_{\aleph} \equiv ((E_i \,\mathsf{sy}\, \mathsf{a}) \Box \psi) \,\mathsf{sy}\, A^i_{\aleph}.$$

It then follows that since ψ contains neither a nor $\widehat{a}$, by $\mathsf{PROP3}$ and $\mathsf{EX2}$, we may rewrite the last expression to $(E_i \Box \psi) \,\mathsf{sy}\, A^i_{\aleph} \,\mathsf{sy}\, \mathsf{a}$.

Case 3: $E_i = [F * G * H]$. Similar to Case 2.

Case 4: $E_i = F \| G$. If $a \in \mathsf{max}_{E_i \,\mathsf{sy}\, A^i_{\aleph}}$ then we proceed as in Case 2. Otherwise, by propositions 6.6 and 6.7(2), we have

$$a \in \mathsf{max}_{F \,\mathsf{sy}\, A^i_{\aleph}} \cap \mathsf{max}_{G \,\mathsf{sy}\, A^i_{\aleph}} \cap \mathsf{covmix}^{A^i_{\aleph}}_{FG}$$

and (without loss of generality and by applying STR2 and STR3, if necessary) $a \in \mathsf{ex}_F$, $\widehat{a} \in \mathsf{ex}_G$ and $\psi = \imath \,\square\, \phi$. By the induction hypothesis and PROP3, $F\,\mathsf{sy}\,A^i_\aleph \equiv F\,\mathsf{sy}\,\mathsf{a}\,\mathsf{sy}\,A^i_\aleph$ and $G\,\mathsf{sy}\,A^i_\aleph \equiv G\,\mathsf{sy}\,\mathsf{a}\,\mathsf{sy}\,A^i_\aleph$. Hence, by PROP6 and INT2,

$$(F\|G)\,\mathsf{sy}\,A^i_\aleph \,\square\, \imath \equiv ((F\,\mathsf{sy}\,A^i_\aleph)\|(G\,\mathsf{sy}\,A^i_\aleph))\,\mathsf{sy}\,A^i_\aleph \,\square\, \imath \equiv ((F\,\mathsf{sy}\,\mathsf{a})\|(G\,\mathsf{sy}\,\mathsf{a}))\,\mathsf{sy}\,A^i_\aleph \,\square\, \imath \equiv (F\|G)\,\mathsf{sy}\,A^i_\aleph\,\mathsf{sy}\,\mathsf{a}.$$

We then proceed as in Case 2 to show that $H_i\,\mathsf{sy}\,A^i_\aleph \equiv ((F\|G) \,\square\, \phi)\,\mathsf{sy}\,A^i_\aleph\,\mathsf{sy}\,\mathsf{a}$.

We have shown that for every $i \leq n$, $H_i\,\mathsf{sy}\,A^i_\aleph \equiv K\,\mathsf{sy}\,\mathsf{a}$, for some K. Thus, by PROP3, $H_i\,\mathsf{sy}\,A^i_\aleph \equiv H_i\,\mathsf{sy}\,A^i_\aleph\,\mathsf{sy}\,\mathsf{a}$, for every $i \leq n$. The last part of the proof is carried out assuming that $n = 2$ (the argument extends easily to $n > 2$).

Assume $J = ((H_1\,\mathsf{sy}\,A_1) \,\square\, (H_2\,\mathsf{sy}\,A_2))\,\mathsf{sy}\,B$. We have shown that $H_i\,\mathsf{sy}\,A_i \equiv H_i\,\mathsf{sy}\,A_i\,\mathsf{sy}\,\mathsf{a}$, for $i = 1, 2$. Moreover, as we have already observed, $B \in \mathsf{simex}_{H_1\,\mathsf{sy}\,A_1, H_2\,\mathsf{sy}\,A_2}$. Hence, by proposition 6.5,

$$a \in \mathsf{covint}^B_{H_1\,\mathsf{sy}\,A_1, H_2\,\mathsf{sy}\,A_2} \cap \mathsf{covnoex}^B_{H_1\,\mathsf{sy}\,A_1, H_2\,\mathsf{sy}\,A_2}.$$

We also have, by proposition 6.8(3,5), that

$$B \in \mathsf{simex}_{H_1\,\mathsf{sy}\,A_1\,\mathsf{sy}\,\mathsf{a}, H_2\,\mathsf{sy}\,A_2\,\mathsf{sy}\,\mathsf{a}} \quad \text{and} \quad a \in \mathsf{covnoex}^B_{H_1\,\mathsf{sy}\,A_1\,\mathsf{sy}\,\mathsf{a}, H_2\,\mathsf{sy}\,A_2\,\mathsf{sy}\,\mathsf{a}}.$$

Hence we can apply LIFT1, in the following way

$$J \;\equiv\; ((H_1\,\mathsf{sy}\,A_1\,\mathsf{sy}\,\mathsf{a}) \,\square\, (H_2\,\mathsf{sy}\,A_2\,\mathsf{sy}\,\mathsf{a}))\,\mathsf{sy}\,B \;\equiv\; ((H_1\,\mathsf{sy}\,A_1) \,\square\, (H_2\,\mathsf{sy}\,A_2))\,\mathsf{sy}\,B\,\mathsf{sy}\,\mathsf{a}$$

which completes the proof. □

9.2 De-synchronisation

We now introduce an auxiliary operator on nets which can be thought of as an 'inverse' of the synchronisation operator, or *de-synchronisation.* For a labelled net $\mathcal{N}$ and a synchronisation set A, we denote by $\mathcal{N}\,\mathsf{unsy}\,A$ the net obtained from $\mathcal{N}$ by deleting all the $\imath$–labelled transitions t for which there are A–synchronisable transitions u and w such that $t \bowtie_{\mathcal{N}} uw$.

Proposition 9.5 Let $\mathcal{N}$ and $\mathcal{M}$ be labelled nets, and A and B be synchronisation sets.

1. $\mathcal{N}\,\mathsf{unsy}\,\emptyset = \mathcal{N}$.
2. If $\mathcal{N} \simeq \mathcal{M}$ then $\mathcal{N}\,\mathsf{unsy}\,A \simeq \mathcal{M}\,\mathsf{unsy}\,A$.
3. If $A \subseteq \mathsf{max}_{\mathcal{N}}$ then $\mathcal{N} \simeq (\mathcal{N}\,\mathsf{unsy}\,A)\,\mathsf{sy}\,A$.
4. If $A \subseteq B$ then $(\mathcal{N}\,\mathsf{sy}\,A)\,\mathsf{unsy}\,B = \mathcal{N}\,\mathsf{unsy}\,B$.
5. If $\mathcal{N}$ is a pre-box then $\mathcal{N}\,\mathsf{unsy}\,A$ is also a pre-box.
6. The $\mathsf{unsy}\,A$ operator distributes over the sequence, parallel and iteration composition.

Proof. (1), (4) and (5) follow directly from the definitions of sy and unsy.

(2) Suppose that $\mathcal{N} \simeq_h \mathcal{M}$. We observe that if $t \simeq_h u$ then $t \in T_{\mathcal{N}\,\mathsf{unsy}\,A}$ if and only if $u \in T_{\mathcal{M}\,\mathsf{unsy}\,A}$ which follows directly from proposition 2.3(1). Moreover, it is easy to see that for all nodes n and n' in $\mathcal{N}\,\mathsf{unsy}\,A$, $n \bowtie_{\mathcal{N}} n'$ if and only if $n \bowtie_{\mathcal{N}\,\mathsf{unsy}\,A} n'$. And, similarly, for all nodes m and m' in $\mathcal{M}\,\mathsf{unsy}\,A$, $m \bowtie_{\mathcal{M}} m'$ if and only if $m \bowtie_{\mathcal{M}\,\mathsf{unsy}\,A} m'$. One can then show that

$$g = \{([n]^{\mathcal{N}\,\mathsf{unsy}\,A}_{\simeq}, [m]^{\mathcal{M}\,\mathsf{unsy}\,A}_{\simeq}) \mid n \in S_{\mathcal{N}} \cup T_{\mathcal{N}\,\mathsf{unsy}\,A} \wedge n \simeq_h m\}$$

is an isomorphism such that $\mathcal{N}\,\mathsf{unsy}\,A \simeq_g \mathcal{M}\,\mathsf{unsy}\,A$.

(3) Since $\mathcal{N} \simeq \mathcal{N}\,\mathsf{sy}\,A$, it suffices to show that $\mathcal{N}\,\mathsf{sy}\,A \simeq (\mathcal{N}\,\mathsf{unsy}\,A)\,\mathsf{sy}\,A$. The latter follows directly from proposition 2.4(2) and the definitions of sy and unsy (essentially, $\mathcal{N}\,\mathsf{sy}\,A$ is $(\mathcal{N}\,\mathsf{unsy}\,A)\,\mathsf{sy}\,A$ with possibly duplicates of some $\imath$–labelled transitions added).

(6) It suffices to observe that, by proposition 2.2, if $\mathcal{M}$ is any of the nets ($\mathcal{N}_i$ or $\mathcal{M}_i$) used in the definition of sequence, concurrent or iteration composition $\mathcal{P}$, and t is a transition in $\mathcal{M}$, then there are no transitions, u and w, of which at least one belongs to the remaining nets forming $\mathcal{P}$, such that $t \bowtie_{\mathcal{P}} uw$. □

De-synchronising a box does not necessarily yield a box. For example, if $E = (((a \,\square\, b); b) \| (\widehat{a} \,\square\, \widehat{b}))\,\mathsf{sy}\,b$ then $\mathsf{box}(E)\,\mathsf{unsy\,a}$ is the net in figure 7 which, as one can easily see, is neither a box nor is duplication equivalent to any box. But what we can say about a de-synchronised box is that it is a box with some transitions added in a way which resembles 'local' synchronisation.

Proposition 9.6 Let $\mathcal{B}$ be a box, A be a synchronisation set and $\mathcal{C} = [\mathcal{B}\,\mathsf{unsy}\,A]_{\simeq}$. Then there is a box $\mathcal{D}$ generated by a synchronisation-free expression and a set of $\imath$-labelled transitions T of $\mathcal{C}$ such that $\mathcal{D}$ is isomorphic to $\mathcal{C}$ with the transitions T deleted. Moreover, for every transition $t \in T$ there are transitions $u, w \in T_{\mathcal{C}} - T$ such that $t \bowtie_{\mathcal{C}} uw$.

Proof. Let $\mathcal{B} = \mathsf{box}(E)$. We first observe that if we omit all the transitions which resulted from synchronisations and then prove the result for the modified box, then the proposition will also hold for the original box. In other words, it suffices to show the result assuming that E does not contain any application of the synchronisation operator. Then, one can show that $\mathcal{C}$ is isomorphic to $\mathsf{box}(F)$, where F is the expression obtained from E by removing, from each subexpression $G = G_1 \,\square\, \ldots \,\square\, G_k$ all the expressions $G_i = \beta$ such that there is $j < i$ satisfying $G_j = G_i$ and, moreover, if $\mathsf{potex}_G \cap A \neq \emptyset$ then also all the expressions of the form $G_i = \imath$ (the proof can be carried out by induction on the structure of E and using proposition 9.5(6)). □

An *ex-path* of a labelled net $\mathcal{N}$ is a sequence of nodes $\pi = n_0 \ldots n_k$ such n_0 is an entry place, n_k is an exit place, and $n_{i-1} \in {}^\bullet n_i$, for every $i \leq k$. We say that $\mathcal{N}$ is *ex-connected*, if for every place s in $\mathcal{N}$ there is an *ex–path* to which s belongs. The following is an easy corollary of the last result and the fact proved in [9] that boxes generated by synchronisation-free expressions are *ex*-connected.

Corollary 9.7 If $\mathcal{B}$ is a box and A is a synchronisation set then $\mathcal{B}\,\mathsf{unsy}\,A$ is *ex*-connected. □

De-synchronisation distributes over the sequence, parallel and iteration composition. However, this does not extend to the choice operator. For example, if $\mathcal{B} = \mathsf{box}(a \| \widehat{a})$ and $\mathcal{C} = \mathsf{box}(b \| \widehat{b})$ then

$$((\mathcal{B}\,\mathsf{sy\,a}) \,\square\, (\mathcal{C}\,\mathsf{sy\,b}))\,\mathsf{unsy\,a} \;=\; \mathcal{B} \,\square\, \mathcal{C} \;\not\simeq\; \mathcal{B} \,\square\, (\mathcal{C}\,\mathsf{sy\,b}) \;=\; ((\mathcal{B}\,\mathsf{sy\,a})\,\mathsf{unsy\,a}) \,\square\, ((\mathcal{C}\,\mathsf{sy\,b})\,\mathsf{unsy\,a}).$$

An important case when de-synchronisation distributes over choice is provided by the next result.

Proposition 9.8 Let $\mathcal{B}_1, \ldots, \mathcal{B}_k$ and $\mathcal{B} = \mathcal{B}_1 \,\square\, \ldots \,\square\, \mathcal{B}_k$ be boxes in Box_0 and A be a synchronisation set such that, for every $i \leq k$, $A \subseteq \mathsf{max}_{\mathcal{B}_i}$ and if $\mathsf{ex}_{\mathcal{B}_i} \neq \emptyset$ then $\mathcal{B}_i = \mathsf{box}(\beta)$ for some action β. Then

$$\mathcal{B}\,\mathsf{unsy}\,A \;=\; (\mathcal{B}_1\,\mathsf{unsy}\,A) \,\square\, \ldots \,\square\, (\mathcal{B}_k\,\mathsf{unsy}\,A).$$

Proof. To show the result it suffices to prove that if $t \in T_{\mathcal{B}_i}$ and there are A-synchronisable transitions u and w in $\mathcal{B}$ satisfying $t \bowtie_{\mathcal{B}} uw$, then $u, w \in T_{\mathcal{B}_i}$. We consider two cases.

Case 1: $\mathsf{ex}_{\mathcal{B}_i} = \emptyset$. If $u, w \notin T_{\mathcal{B}_i}$ then we obtain a contradiction with propositions 2.2 and 5.2(2) and $t \notin \mathsf{EX}_{\mathcal{B}}$ and $\mathcal{B}_i \in \mathsf{Box}_0$. If $u \in T_{\mathcal{B}_i}$ and $w \notin T_{\mathcal{B}_i}$ then, by proposition 2.2, $w \in \mathsf{EX}_{\mathcal{B}}$. Hence $t \bowtie_{\mathcal{B}_i} u\delta$ and from proposition 5.2(1) it follows that $\mathsf{EX}_{\mathcal{B}_i} \neq \emptyset$, a contradiction. Thus $u, w \in T_{\mathcal{B}_i}$.

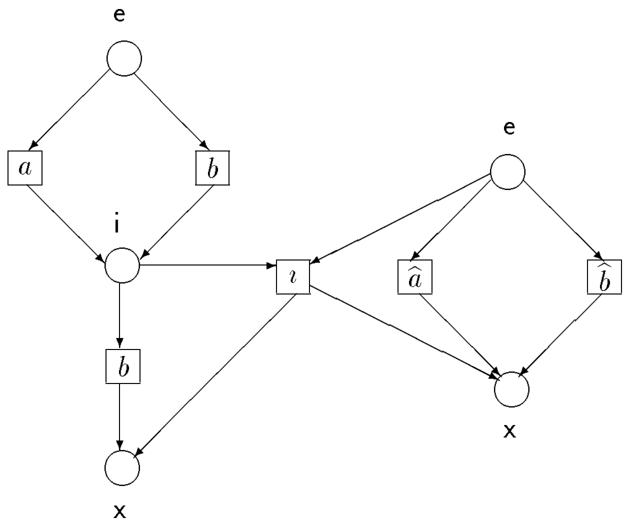


Figure 7: De-synchronised box may not be a box.

Case 2: $\mathsf{ex}_{\mathcal{B}_i} \neq \emptyset$. Then $\mathcal{B}_i = \mathsf{box}(\imath)$ and, using proposition 2.2, one can easily see that both u and w belong to the same $\mathcal{B}_j$. Hence, by $A \subseteq \mathsf{max}_{\mathcal{B}_j}$ and $uw \bowtie_{\mathcal{B}_j} \delta$, we have $\mathsf{ex}_{\mathcal{B}_j} \neq \emptyset$, a contradiction. $\square$

The next definition and the proposition that follows deal with the problem of a unique representation of a net as a composition of other, smaller, nets. Below, a pre-box $\mathcal{B}$ is *choice-decomposable* (*sequence-decomposable*) if there are pre-boxes $\mathcal{D}$ and $\mathcal{H}$ such that $\mathcal{B} \simeq \mathcal{D} \,\square\, \mathcal{H}$ (resp. $\mathcal{B} \simeq \mathcal{D}; \mathcal{H}$). Note that $\mathsf{box}(\beta)$ is choice-decomposable, for every action β. An *iteration / parallel / sequence / choice decomposition* of a pre-box $\mathcal{B}$ is a sequence of pre-boxes $\mathcal{B}_1, \ldots, \mathcal{B}_k$ such that, respectively, the following hold:

- $k = 3$ and $\mathcal{B} \simeq [\mathcal{B}_1 * \mathcal{B}_2 * \mathcal{B}_3]$.
- $k \geq 2$ and $\mathcal{B} \simeq \mathcal{B}_1 \| \ldots \| \mathcal{B}_k$ and, for every $i \leq k$, $\mathcal{B}_i$ is connected.
- $k \geq 2$ and $\mathcal{B} \simeq \mathcal{B}_1; \ldots; \mathcal{B}_k$ and, for every $i \leq k$, $\mathcal{B}_i$ is not sequence-decomposable.
- $k \geq 2$ and $\mathcal{B} \simeq \mathcal{B}_1 \,\square\, \ldots \,\square\, \mathcal{B}_k$ and, for every $i \leq k$, if $\mathcal{B}_i$ is choice-decomposable then $\mathcal{B}_i \simeq \mathsf{box}(\beta)$ for some action β and $\mathcal{B}_i \not\simeq \mathcal{B}_j$ for all $j \neq i$.

Proposition 9.9 Let $\mathcal{B}$ be a box and A be a synchronisation set.

1. $\mathcal{B}\,\mathsf{unsy}\,A$ has at most one (up to duplication equivalence and permutation) parallel decomposition and choice decomposition.
2. $\mathcal{B}\,\mathsf{unsy}\,A$ has at most one (up to duplication equivalence) sequence decomposition and iteration decomposition.

Proof. This proposition is proved using the results obtained in [9]. We here give a sketch of how this can be done for the case of sequence decomposition.

To start with, one can show that if $\mathcal{G}_1, \ldots, \mathcal{G}_k$ is a sequence decomposition of a pre-box $\mathcal{G}$ then $[\mathcal{G}_1]_\simeq, \ldots, [\mathcal{G}_k]_\simeq$ is a sequence decomposition of $[\mathcal{G}]_\simeq$ such that $[\mathcal{G}]_\simeq = [\mathcal{G}_1]_\simeq; \ldots; [\mathcal{G}_k]_\simeq$. Hence it suffices to show that $\mathcal{C} = [\mathcal{B}\,\mathsf{unsy}\,A]_\simeq$ has at most one sequence decomposition $\mathcal{B}_1, \ldots, \mathcal{B}_k$ such that $\mathcal{C} = \mathcal{B}_1; \ldots; \mathcal{B}_k$.

By proposition 9.6, we may assume that there is a box $\mathcal{D}$ generated by a synchronisation-free expression E (for which the results obtained in [9] can be applied) and transitions t, u, w in $\mathcal{C}$ such that $t \bowtie_{\mathcal{C}} uw$, $(u, w) \in \mathsf{syn}_A$ and $\mathcal{D}$ is $\mathcal{C}$ with t deleted (i.e. we assume that $|T| = 1$ where T is as in the formulation of proposition 9.6; the argument is similar for $|T| > 1$). Then the following hold:

(i) Every representation of $\mathcal{C}$ as a sequential composition of pre-boxes induces a corresponding representation of $\mathcal{D}$. More precisely, if we have $\mathcal{C} = \mathcal{B}_1; \ldots; \mathcal{B}_k$, for some pre-boxes $\mathcal{B}_i$, and $t \in T_{\mathcal{B}_l}$ then $\mathcal{D} = \mathcal{D}_1; \ldots; \mathcal{D}_k$ where $\mathcal{D}_l$ is $\mathcal{B}_l$ with t deleted, and $\mathcal{D}_i = \mathcal{B}_i$, for all $i \neq l$.

(ii) It was shown in [9] that there is a unique sequence of boxes $\mathcal{F}_1, \ldots, \mathcal{F}_m$ for $\mathcal{D}$ such that whenever $\mathcal{D} = \mathcal{D}_1; \ldots; \mathcal{D}_l$ there are (unique) integers $0 = m_0 < m_1 < \ldots < m_l = m$ such that, for every $i \leq l$, $\mathcal{D}_i = \mathcal{F}_{1+m_{i-1}}; \ldots; \mathcal{F}_{m_i}$ (in particular, this means that no F_i is sequence-decomposable). This and (i) implies that for $\mathcal{C}$ there is at most one sequence of non-sequence-decomposable pre-boxes $\mathcal{B}_1, \ldots, \mathcal{B}_k$ such that $\mathcal{C} = \mathcal{B}_1; \ldots; \mathcal{B}_k$ (basically, if $u \in T_{\mathcal{F}_x}$ and $w \in T_{\mathcal{F}_y}$ $(x \leq y)$ then $\mathcal{B}_1, \ldots, \mathcal{B}_k$ is $\mathcal{F}_1, \ldots, \mathcal{F}_{x-1}, (\mathcal{F}_x; \mathcal{F}_{x+1}; \ldots; \mathcal{F}_y), \mathcal{F}_{y+1}, \ldots, \mathcal{F}_m)$. □

9.3 Normal form box expressions

A box expression E in Exp_0 is in *normal form* if it is in one of the following five types. Below, $A = \mathsf{max}_E$ and each E_i is an expression in normal form such that $A \subsetneq \mathsf{max}_{E_i}$. Moreover, $\mathcal{B}$ denotes $\mathsf{box}(E)$ unsy A and $\mathcal{B}_i$ denotes $\mathsf{box}(E_i)$ unsy A.

- Type-a $\quad E = \beta$ for some action β.
- Type-i $\quad E = [E_1 * E_2 * E_3]$ sy A.
- Type-p $\quad E = (E_1 \| \ldots \| E_k)$ sy A and $\mathcal{B}_1, \ldots, \mathcal{B}_k$ is a parallel decomposition of $\mathcal{B}$.
- Type-c $\quad E = (E_1 \,\square\, \ldots \,\square\, E_k)$ sy A and $\mathcal{B}_1, \ldots, \mathcal{B}_k$ is a choice decomposition of $\mathcal{B}$.
- Type-s $\quad E = (E_1; \ldots; E_k)$ sy A and $\mathcal{B}_1, \ldots, \mathcal{B}_k$ is a sequence decomposition of $\mathcal{B}$.

We now aim at showing that duplication equivalent expressions in normal form are equal (up to permutation of subexpressions in choice and parallel composition contexts). The first step is to show that any two duplication equivalent expressions are of the same type. The proof will rely on two properties of labelled nets, called internal connectedness and internal interface, introduced in [9]. Note that the application of the results based on internal connectedness and internal interfaces from [9] is possible due to corollary 9.7. The relevant proofs from [9] can easily be adapted and we therefore omit here their detailed exposition, referring instead to the appropriate parts of [9].

A pre-box $\mathcal{B}$ is *internally connected* if it is connected after removing all the entry and exit places. It has an *internal interface* if there is a set of internal places P such that if we delete P then $\mathcal{B}$ can be divided into two disjoint subgraphs with the nodes N_1 and N_2 such that: (i) each node in N_1 is connected to an entry place and not connected to any exit place; (ii) each node in N_2 is connected to an exit place and not connected to any entry place; and (iii) if we take $\mathcal{B}_i$ to be $\mathcal{B}$ with the nodes N_i deleted $(i = 1, 2)$, then P is a $\otimes$-set for $\mathcal{B}_1$ and $\mathcal{B}_2$. For example, $P = \{s_2, s_3\}$ is the only internal interface of the net in figure 1.

Proposition 9.10 Let E be an expression in normal form and m be the number of transitions in $\mathcal{B} = \mathsf{box}(E)$ unsy max_E.

1. If E is of type-a then $m = 1$.
2. If E is of type-p then $m > 1$ and $\mathcal{B}$ is not connected.

3. If E is of type-c then $m > 1$ and $\mathcal{B}$ is connected and not internally connected.

4. If E is of type-i then $m > 1$ and $\mathcal{B}$ is connected and internally connected and has no internal interface.

5. If E is of type-s then $m > 1$ and $\mathcal{B}$ is connected and internally connected and has at least one internal interface.

Proof. Let the $\mathcal{B}$'s and k be as in the definition of normal form. Note that, by proposition 9.5(5) and corollary 9.7, each $\mathcal{B}_i$ is an ex-connected pre-box.

(1) obviously holds as well as $m > 1$ in (2)–(5).

(2) Then $\mathcal{B} \simeq \mathcal{B}_1 \| \ldots \| \mathcal{B}_k$. This and $k \geq 2$ means that $\mathcal{B}$ is not connected.

(3) Then $\mathcal{B} \simeq \mathcal{B}_1 \Box \ldots \Box \mathcal{B}_k$. Since $k \geq 2$ and each $\mathcal{B}_i$ is ex-connected, it follows that $\mathcal{B}$ is connected but not internally connected (c.f. Lemma 5 in [9]).

(4) Then $\mathcal{B} \simeq [\mathcal{B}_1 * \mathcal{B}_2 * \mathcal{B}_3]$. Since each $\mathcal{B}_i$ is ex-connected, it follows that $\mathcal{B}$ is connected, internally connected, and has no internal interface (c.f. Lemmata 5 and 7 in [9]).

(5) Then $\mathcal{B} \simeq \mathcal{B}_1; \ldots; \mathcal{B}_k$. Since $k \geq 2$ and each $\mathcal{B}_i$ is ex-connected, it follows that $\mathcal{B}$ is connected, internally connected, and has an internal interface (c.f. Lemmata 5 and 7 in [9]). $\Box$

Directly from propositions 9.5(2) and 9.10, and the fact that connectedness, internal connectedness and having an internal interface are all net properties preserved by duplication equivalence, we obtain that

Corollary 9.11 Duplication equivalent normal form expressions are of the same type. $\Box$

Moreover, one can show that duplication equivalent normal form expressions are equal.

Proposition 9.12 If E and F are duplication equivalent expressions in normal form then $E = F$ up to permutation of the components in subexpressions of the form $E_1 \| \ldots \| E_k$ and $E_1 \Box \ldots \Box E_k$.

Proof. Clearly, $\mathsf{max}_E = \mathsf{max}_F = A$ and, by corollary 9.11, both E and F are of the same type. The proof proceeds by induction on the structure of E. The base case is $E = \beta$ and $F = \beta'$ which clearly implies $E = F$.

In the inductive step, suppose that E and F are of type-p and, moreover, that $E = (E_1 \| \ldots \| E_k)\, \mathsf{sy}\, A$ and $E = (F_1 \| \ldots \| F_m)\, \mathsf{sy}\, A$. By proposition 9.5(2), $\mathsf{box}(E)\, \mathsf{unsy}\, A \simeq \mathsf{box}(F)\, \mathsf{unsy}\, A$. Hence, by proposition 9.9(1), $k = m$ and, without loss of generality (one can always use $\mathsf{STR4}$ and $\mathsf{STR5}$), $\mathsf{box}(E_i)\, \mathsf{unsy}\, A \simeq \mathsf{box}(F_i)\, \mathsf{unsy}\, A$ and so $\mathsf{box}(E_i)\, \mathsf{unsy}\, A\, \mathsf{sy}\, A \simeq \mathsf{box}(F_i)\, \mathsf{unsy}\, A\, \mathsf{sy}\, A$, for every $i \leq k$. Thus, by $A \subseteq \mathsf{max}_{E_i} = \mathsf{max}_{F_i}$ and proposition 9.5(3), we obtain $E_i \simeq F_i$, for every $i \leq k$. Now, since both E_i and F_i are in normal form, by the induction hypothesis, $E_i = F_i$ (up to permutation of subexpressions in choice and parallel contexts). Hence the proposition holds. The proofs for E of type-c, type-i and type-s are similar. $\Box$

The next proposition is an auxiliary result used later, in proposition 9.14, to transform an expression into a normal form expression.

Proposition 9.13 Let E be a normal form expression and $A \subseteq \mathsf{max}_E$ be a synchronisation set.

1. If E is of type-p and $\mathsf{box}(E)\, \mathsf{unsy}\, A$ is not connected then there is $F = (F_1 \| \ldots \| F_k)\, \mathsf{sy}\, A$ $(k \geq 2)$ such that $E \equiv F$ and, for every $i \leq k$, $A \subseteq \mathsf{max}_{F_i}$ and $\mathsf{box}(F_i)\, \mathsf{unsy}\, A$ is connected.

2. If E is of type-s and $\mathsf{box}(E)\, \mathsf{unsy}\, A$ is choice decomposable then there is an expression $F = (F_1; \ldots; F_k)\, \mathsf{sy}\, A$ $(k \geq 2)$ such that $E \equiv F$ and, for every $i \leq k$, $A \subseteq \mathsf{max}_{F_i}$ and $\mathsf{box}(F_i)\, \mathsf{unsy}\, A$ is not sequence-decomposable.

3. If E is of type-c, $\mathsf{ex}_E = \emptyset$ and $\mathsf{box}(E)\,\mathsf{unsy}\,A$ is choice decomposable then there is an expression $F = (F_1 \square \ldots \square F_k)\,\mathsf{sy}\,A$ ($k \geq 2$) such that $E \equiv F$ and, for every $i \leq k$, $A \subseteq \mathsf{max}_{F_i}$ and $\mathsf{box}(F_i)\,\mathsf{unsy}\,A$ is not choice-decomposable.

Proof. (1) Suppose $E = (E_1\|\ldots\|E_m)\,\mathsf{sy}\,B$. Define a relation ρ on $\{1,\ldots,m\}$ in such a way that $(i,j) \in \rho$ if $i \neq j$ and

$$(E_i\|E_j)\,\mathsf{sy}\,B \not\simeq (E_i\|E_j)\,\mathsf{sy}\,A. \tag{4}$$

From the definition of type-p and $A \subseteq B$ it follows that $E_l \simeq E_l\,\mathsf{sy}\,B \simeq E_l\,\mathsf{sy}\,A$ and $\mathsf{box}(E_l)\,\mathsf{unsy}\,A$ is connected, for every $l \leq m$. Hence $(i,j) \in \rho$ if and only if $i \neq j$ and $\mathsf{box}((E_i\|E_j)\,\mathsf{sy}\,B)\,\mathsf{unsy}\,A$ is a connected pre-box. The graph of the relation ρ can be divided into k connected components. Without loss of generality (one can always use STR4 and STR5) we may assume that $0 = m_0 < m_1 < \ldots < m_k = m$ are such that, for every $i \leq k$, the integers $1 + m_{i-1}, \ldots, m_i$ are the vertices of the i-th connected component of the graph of ρ. Denote $G_i = F_{1+m_{i-1}}\|\ldots\|F_{m_i}$, for every $i \leq k$. Clearly, for every $i \leq k$, $\mathsf{box}(G_i)\,\mathsf{sy}\,B\,\mathsf{unsy}\,A$ is connected. Thus, since $\mathsf{box}(E)\,\mathsf{unsy}\,A$ is not connected, we have $k \geq 2$. We then observe that from proposition 5.3, corollary 3.9 and $A \subseteq B$, it follows that

$$((G_1\,\mathsf{sy}\,B)\|\ldots\|(G_k\,\mathsf{sy}\,B))\,\mathsf{sy}\,B \simeq ((G_1\,\mathsf{sy}\,B)\|\ldots\|(G_k\,\mathsf{sy}\,B))\,\mathsf{sy}\,A.$$

Denote $F_i = G_i\,\mathsf{sy}\,B$, for every $i \leq k$, and $F = (F_1\|\ldots\|F_k)\,\mathsf{sy}\,A$. We have $E \equiv (F_1\|\ldots\|F_k)\,\mathsf{sy}\,B \simeq (F_1\|\ldots\|F_k)\,\mathsf{sy}\,A$. Hence, by proposition 9.4 and PROP3, $E \equiv F$.

(2,3) We proceed similarly as in the case of type-p expression. The main difference is that in the case of type-s expression, $0 = m_0 < m_1 < \ldots < m_k = m$ are assumed to be integers such that, for every $i \leq k$, there is a path in the graph of ρ between $1 + m_{i-1}$ and m_i and, for every $1 \leq j < k$, there is no path between x and y if $x \leq m_i < y$. □

Not every expression in Exp_0 can be rewritten into a normal form box expression. For example, if $E = ((a;\imath) \square \widehat{a})\,\mathsf{sy}\,\mathsf{a} \square (\imath; a)$ then $\mathsf{max}_E = \mathcal{A} - \mathsf{a}$ and the only decomposition of $\mathsf{box}(E)\,\mathsf{unsy}\,(\mathcal{A} - \mathsf{a}) = \mathsf{box}(E)$ into boxes $\mathcal{B}_i$ which could satisfy one of the parts of the definition of normal form expression, are the three nets shown in figure 8 (note that $\mathsf{box}(E)\,\mathsf{unsy}\,(\mathcal{A} - \mathsf{a}) = \mathcal{B}_1 \square \mathcal{B}_2 \square \mathcal{B}_3$). While the first two nets do not create any problems, the third one does, as it is easy to see that there is no box expression E_3 such that $\mathcal{B}_3$ is duplication equivalent to $\mathsf{box}(E_3)\,\mathsf{unsy}\,A$, for any synchronisation set A. Hence E has no normal form in the sense defined above. Therefore we need to restrict the applicability of the choice operator. The definition below is motivated by the way in which the fourth case in the definition of normal form has been formulated (and, indirectly, by the characterisation of the situation when the unsy operator distributes over choice).

A box expression $E \in \mathsf{Exp}_0$ is *choice-restricted* if every subexpression F of E which has choice as the topmost operator is of the form $\beta_1 \square \ldots \square \beta_k \square H$ and satisfies $\mathsf{ex}_{H\,\mathsf{sy}\,A} = \emptyset$, where A is the union of all the synchronisation sets B such that F lies within the scope of an application of $\mathsf{sy}\,B$ (we allow $k = 0$, and H may be missing if $k > 0$).[7] Then, we denote by Exp_1 the set of all box expressions $G \in \mathsf{Exp}_0$ such that $G \equiv E$, for some choice-restricted box expression E. Note that the definition of Exp_1 is not fully syntactic. However, one can give simple syntactic conditions which guarantee that a box expression which is not choice-restricted belongs to Exp_1.

Proposition 9.14 If E is an expression in Exp_1 then there is an expression in normal form F such that $E \equiv F$.

Proof. We may assume that E is choice-restricted, and then proceed by induction on the number of transitions in the duplication quotient of $\langle E \rangle$. The base case is $\langle E \rangle = \beta$ or $\langle E \rangle = \beta \square \ldots \square \beta$, for

[7] Note that H can be an expression whose main connective is choice.

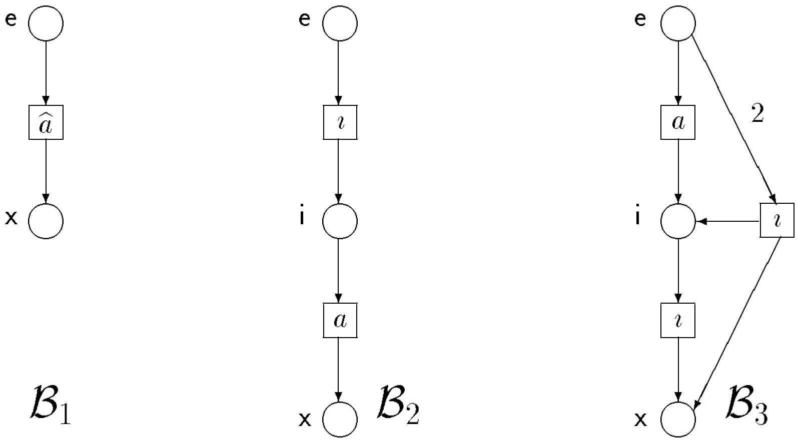


Figure 8: Decomposition of a de-synchronised box.

some action β. Then $E \equiv \beta$ or $E \equiv \beta \,\square\, \ldots \,\square\, \beta$ which can be shown using PROP1, PROP2, PROP3 and PROP5. Moreover, in the latter case, using DUPL, STR2 and STR3, we may transform $\beta \,\square\, \ldots \,\square\, \beta$ into β. In the inductive step we now consider a number of cases, depending on the form of $\langle E \rangle$.

Case 1: $\langle E \rangle = E' \| E''$. By PROP3, PROP6, and proposition 9.4, we have $E \equiv (F \,\mathsf{sy}\, A \| G \,\mathsf{sy}\, A) \,\mathsf{sy}\, A$, where $A = \mathsf{max}_E$ and $F, G \in \mathsf{Exp}_1$. By the induction hypothesis, $E \equiv (F_0 \| G_0) \,\mathsf{sy}\, A$, where $F_0 \equiv F \,\mathsf{sy}\, A$ and $G_0 \equiv G \,\mathsf{sy}\, A$, and both F_0 and G_0 are in normal form. If both $\mathsf{box}(F_0) \,\mathsf{unsy}\, A$ and $\mathsf{box}(G_0) \,\mathsf{unsy}\, A$ are connected, then we are done since, by proposition 9.5(2,4,6),

$$\begin{array}{rcl} \mathsf{box}(E) \,\mathsf{unsy}\, A & \simeq & \mathsf{box}((F_0 \| G_0) \,\mathsf{sy}\, A) \,\mathsf{unsy}\, A \\ & = & \mathsf{box}(F_0 \| G_0) \,\mathsf{unsy}\, A = (\mathsf{box}(F_0) \,\mathsf{unsy}\, A) \| (\mathsf{box}(G_0) \,\mathsf{unsy}\, A). \end{array}$$

Otherwise, without loss of generality, we assume that $\mathsf{box}(F_0) \,\mathsf{unsy}\, A$ is not connected and $G_0 \,\mathsf{unsy}\, A$ is connected. Then, by proposition 9.13(1), $F_0 \equiv (F_1 \| \ldots \| F_k) \,\mathsf{sy}\, A$ where $k \geq 2$, $A \subseteq \mathsf{max}_{F_i}$ and $\mathsf{box}(F_i) \,\mathsf{unsy}\, A$ is connected, for every $i \leq k$. We now can apply k-times the induction hypothesis to obtain that $E \equiv ((H_1 \| \ldots \| H_k) \,\mathsf{sy}\, A \| G_0) \,\mathsf{sy}\, A$, where each H_i is a normal form expression for every F_i. Next, by PROP3 and PROP5, we obtain $E \equiv (H_1 \| \ldots \| H_k \| G_0) \,\mathsf{sy}\, A$ which completes the discussion of this case since, by proposition 9.5(2,4,6), we have

$$\begin{array}{rcl} \mathsf{box}(E) \,\mathsf{unsy}\, A & \simeq & \mathsf{box}((H_1 \| \ldots \| H_k \| G_0) \,\mathsf{sy}\, A) \,\mathsf{unsy}\, A \\ & = & \mathsf{box}(H_1 \| \ldots \| H_k \| G_0) \,\mathsf{unsy}\, A \\ & = & (\mathsf{box}(H_1) \,\mathsf{unsy}\, A) \| \ldots \| (\mathsf{box}(H_k) \,\mathsf{unsy}\, A) \| (\mathsf{box}(G_0) \,\mathsf{unsy}\, A). \end{array}$$

Case 2: $\langle E \rangle = [E' * E'' * E''']$. By PROP3, PROP6, and proposition 9.4 we have $E \equiv [F \,\mathsf{sy}\, A * G \,\mathsf{sy}\, A * H \,\mathsf{sy}\, A] \,\mathsf{sy}\, A$, where $A = \mathsf{max}_E$ and $F, G, H \in \mathsf{Exp}_1$. By the induction hypothesis, $E \equiv [F_0 * G_0 * H_0] \,\mathsf{sy}\, A$, where F_0, G_0 and H_0 are in normal form and $F_0 \equiv F \,\mathsf{sy}\, A$, $G_0 \equiv G \,\mathsf{sy}\, A$ and $H_0 \equiv H \,\mathsf{sy}\, A$.

Case 3: $\langle E \rangle = E'; E''$. We proceed similarly as in Case 1.

Case 4: $\langle E \rangle = E' \,\square\, E''$. By the definition of Exp_1, PROP3, PROP6, and proposition 9.4, $E \equiv (\beta_1 \,\square\, \ldots \,\square\, \beta_k \,\square\, G \,\mathsf{sy}\, A) \,\mathsf{sy}\, A$, where $G \in \mathsf{Exp}_1$, $A = \mathsf{max}_E$ and $\mathsf{ex}_{G \,\mathsf{sy}\, A} = \emptyset$. Moreover, by DUPL, we may assume that $\beta_i \neq \beta_j$, for every $i \neq j$. By the induction hypothesis, $E \equiv (\beta_1 \,\square\, \ldots \,\square\, \beta_k \,\square\, G_0) \,\mathsf{sy}\, A$, where $G_0 \equiv G \,\mathsf{sy}\, A$ and G_0 is in normal form. If $G_0 \,\mathsf{unsy}\, A$ is not choice-decomposable, then we are done since, by proposition 9.8,

$$\begin{array}{rcl}
\mathsf{box}(E)\ \mathsf{unsy}\ A & \simeq & \mathsf{box}((\beta_1 \square \ldots \square \beta_k \square G_0)\ \mathsf{sy}\ A)\ \mathsf{unsy}\ A \\
& = & \mathsf{box}((\beta_1 \square \ldots \square \beta_k \square G_0))\ \mathsf{unsy}\ A \\
& = & (\mathsf{box}(\beta_1)\ \mathsf{unsy}\ A) \square \ldots \square (\mathsf{box}(\beta_k)\ \mathsf{unsy}\ A) \square (\mathsf{box}(G_0)\ \mathsf{unsy}\ A) \square (\mathsf{box}(G_0)\ \mathsf{unsy}\ A).
\end{array}$$

Otherwise, we can apply proposition 9.13(3), and then proceed similarly as in case 1. $\square$

We can finally prove the main result of the second part of this paper.

Theorem 9.15 For all box expression E, F in Exp_1, if $E \simeq F$ then $E \equiv F$. $\square$

Proof. Follows from propositions 9.12 and 9.14. $\square$

10 Concluding remarks

In this paper we have investigated a notion of equivalence within the Petri Box Calculus defined for the box expressions and based on the structural similarity of the corresponding boxes (labelled Petri nets). In particular, we have developed a sound and complete axiomatisation of duplication equivalence for a subset of box expressions. In doing so, it turned out that a crucial problem to be solved was that of a structural characterisation of maximal synchronisation sets of box expressions. We have found that such a characterisation is rather complicated for box expressions whose main connective (other than synchronisation) is the choice composition. This has led to a restriction on the set of box expressions for which the soundness and completeness results directly apply. The duplication equivalence is a very strong notion of equivalence which resembles the strong equivalence of CCS [15]. It is therefore natural to envisage that the future research will be concentrated on developing an axiomatisation of a weaker equivalence on box expressions, similar to the observational congruence of CCS. From this point of view the results obtained here are highly relevant since any axiomatisation of a weaker behavioural equivalence would encompass the axiomatisation of duplication equivalence. Moreover, the restrictions imposed on the type of box expressions for which the soundness and completeness results hold seem to be rather mild when considering a weaker notion of equivalence. Without going into details, if $F_1, \ldots, F_k$ are the subexpressions of a box expression E which cause the latter not to belong to Exp_1 then it should be possible (under any reasonable notion of observational equivalence which ignores internal moves) to replace each F_i by $F_i; \imath$, within E, and the resulting expression, call it $E^{\langle \imath \rangle}$, would now belong to Exp_1. We also conjecture that the same transformation can be used to extend in a somewhat unusual way the completeness result obtained here, in the following way. If E and F are arbitrary box expressions such that $E \simeq F$ then $E^{\langle \imath \rangle} \equiv F^{\langle \imath \rangle}$, where it is assumed that for a box expression E in Exp_1, $E^{\langle \imath \rangle} = E$.

Our final remark concerns the non-standard way in which some of the axioms were formulated since they refer to various sets (even sets of sets) of actions, such as covall. The reader might question whether this leads to a significant increase in the algorithmic complexity of the axiomatisation developed here when compared, e.g., with that presented in [15]. The answer is that it does not, as it is not difficult to see that all the sets involved are 'small' which is due to an easy observation that it is always the case that

$$\sum_{A \in \mathsf{ccall}_E} |A| \;\leq\; k \quad \text{and} \quad |\mathsf{ex}_E| \;\leq\; k$$

where k is the number of action occurrences in a box expression E.

References

[1] J. Baeten and W.P. Weijland: *Process Algebra.* Cambridge Tracts in Theoretical Computer Science 18 (1990).

[2] E. Best, R. Devillers and J. Hall: *The Petri Box Calculus: a New Causal Algebra with Multilabel Communication.* Advances in Petri Nets 1992, G. Rozenberg (ed.). Springer-Verlag, Lecture Notes in Computer Science Vol. 609, 21–69 (1992).

[3] E. Best and R.P. Hopkins: $B(PN)^2$ *- a Basic Petri Net Programming Notation.* Proc. of PARLE'93, Springer-Verlag, Lecture Notes in Computer Science Vol. 694, 379-390 (1993).

[4] E. Best and M. Koutny: *A Refined View of the Box Algebra.* Proc. Petri Net Conference'95, G. De Michelis, M. Diaz (eds). Springer-Verlag, Lecture Notes in Computer Science Vol. 935, 1–20 (1995).

[5] E. Best and M. Koutny: *Solving Recursive Net Equations.* Proc. ICALP'95, Z. Fülöp and F. Gécseg (eds). Springer-Verlag, Lecture Notes in Computer Science Vol. 944, 605–623 (1995).

[6] R. Devillers: *S-invariant Analysis of Recursive Petri Boxes.* Acta Informatica Vol. 32, 313-345 (1995).

[7] U. Goltz: *On Representing CCS Programs by Finite Petri Nets.* Proc. MFCS'88, Springer-Verlag, Lecture Notes in Computer Science Vol. 324, 339-350 (1988).

[8] M.B. Hennessy: *Algebraic Theory of Processes.* The MIT Press (1988).

[9] M. Hesketh: *On Synthesis of Box Expressions from Petri Boxes.* Technical Report TR-552, Department of Computing Science, University of Newcastle upon Tyne (1996).

[10] C.A.R. Hoare: *Communicating Sequential Processes.* Prentice Hall (1985).

[11] INMOS: *Occam 2 Reference Manual.* Prentice-Hall (1988).

[12] M. Koutny, J. Esparza and E. Best: *Operational Semantics for the Petri Box Calculus.* Proc. CONCUR'94, B. Jonsson and J.Parrow (eds), Springer-Verlag, Lecture Notes in Computer Science Vol. 836, 210-225 (1994).

[13] M. Koutny: *Partial Order Semantics of Box Expressions.* Proc. Petri Net Conference'94, R. Valette (ed.), Springer-Verlag, Lecture Notes in Computer Science Vol. 815, 318-337 (1994).

[14] M. Koutny and E. Best: *Operational and Denotational Semantics for the Box Algebra.* Theoretical Computer Science (to appear).

[15] R. Milner: *Communication and Concurrency.* Prentice Hall (1989).

[16] E.R. Olderog: *Nets, Terms and Formulas.* Cambridge Tracts in Theoretical Computer Science 23 (1991).

[17] W. Reisig: *Petri Nets. An Introduction.* Springer-Verlag, EATCS Monographs on Theoretical Computer Science Vol. 3 (1985).