Order Structures and Generalisations of Szpilrajn's Theorem

Ryszard Janicki* and Maciej Koutny**

Abstract. Relational structures of the form (X, R_1, R_2), with $R_1 \subseteq R_2 \subseteq X \times X$, R_1 being a poset interpreted as causality, R_2 being interpreted as 'not later than' or 'weak causality' relation, are considered. Szpilrajn's theorem that each poset is the intersection of its total extensions is generalised to such structures; the interpretation and applications of the results obtained are discussed.

1 Introduction

A familiar theorem that each partial order is the intersection of the set of its total extensions, due to Szpilrajn (Marczewski since 1940) [20], has found various applications in concurrency theory [3, 18, 19]. Its usual interpretation is that causality relation, understood as a specification of concurrent behaviour, is completely defined by the set of its sequential observations. But concurrent behaviours can be very complex. For instance, if priorities are combined with non-instantaneous events, (causal) partial orders alone may be insufficient to provide a full and adequate specification [4, 9, 11, 13].

In several papers [1, 2, 8, 13, 16] relational structures of the form (X, R_1, R_2), where X is a set of events, $R_1 \subseteq R_2 \subseteq X \times X$ and R_1 is a poset, were proposed and analysed in order to specify more adequately complex concurrent behaviours. R_1 was usually interpreted as 'precedence' or 'causality' while R_2 was called 'weak precedence' or 'not later than' relation. Whereas R_1 was always defined as a poset, the assumed properties of R_2 did vary. They were basically the same in [8] and [14], but different in [16] and in [1, 2] (the axioms for R_2 in [1, 2] were refined version of those formulated in [16]).

In this paper we deal with a more general definition of (X, R_1, R_2), with the structures of [1, 2, 8, 13, 16] being treated as special cases. We show that these new relational structures can be treated as extensions of general partial orders, while the structures of [1, 2] as extensions of interval orders, and the structures of [8, 14] as extensions of stratified orders. We also show how Szpilrajn's theorem can be generalised in each case.

The paper is organised as follows. After recalling some basic results concerning posets and briefly discussing weak causality, in Section 4 we define an *order*

* Department of Computer Science and Systems, McMaster University, Hamilton, Ontario, Canada L8S 4K1. Supported by NSERC grant, No. OGP 0036539.

** Department of Computing Science, University of Newcastle, Newcastle upon Tyne NE1 7RU, U.K. Supported by Esprit Basic Research Working Group 6067 CALIBAN

structure, our central notion, as a direct generalisation of the concept of a partial order. Section 5 is devoted to three special cases, namely interval, stratified and total order structures. The generalisations of Szpilrajn's theorem are formulated and partially proven in Section 6. In Section 7 we provide representation theorems for interval and stratified order structures as well as the rest of the proof of the main theorem from Section 6.

2 Posets and Weak Causality

A *partially ordered set* (poset) is a pair $po = (X, \prec)$ such that X is a set and $\prec$ is an irreflexive transitive relation on X. We will write $a \sim b$ if a and b are distinct incomparable elements of X, and $a \preceq b$ if $a \prec b$ or $a \sim b$. We shall also write $\prec_{po}$ to denote $\prec$, if necessary. The poset is: *total* if $\sim$ is empty; *stratified* [7] (or *weak* [6]) if $\sim \cup\, id_X$ is an equivalence relation; and *interval* [6] if $a \prec b \wedge c \prec d \;\Rightarrow\; a \prec d \vee c \prec b$.

Theorem 1 (Szpilrajn's Theorem). *For every poset* $(X, \prec)$, $\prec\, = \bigcap_{q \in Q} \prec_q$, *where* Q *is the set of all total orders* $(X, \prec_q)$ *such that* $\prec \subseteq \prec_q$.

In the concurrency theory posets are used to model both specifications and observations of behaviours. Specifications usually involve a kind of *causality* relation [3] which is modelled by general posets, whereas observations have been modelled by various types of posets, e.g., total, stratified, interval, as well as general ones. The most restricted case involves sequential observers who can only observe sequential universum; their observations are modelled by total orders of event occurrences. This model is sufficient if events are regarded as instantaneous. Note that in such a case two concurrent events, a and b, can be observed either as a sequence ab or as ba. If events cannot be regarded as instantaneous, recording simultaneous observations of events is necessary. When simultaneity is assumed to be transitive, observations are modelled by stratified orders (step-sequences in the finite case). A more general model of observations is based on the concept of interval order. Total and stratified orders have been used to model observations for years. Recently, many authors advocate and show advantages of using interval orders [10, 12, 17, 21]. [12, 15] provide a detailed motivation and rigorous mathematical treatment for this approach from the viewpoint of concurrency theory. The general theory of interval orders can be found in the monograph [6], while the concept itself can be traced back to Wiener's 1914 paper [22]. [18] allows observers to work as teams, which gives them the power to observe general posets.

As we already mentioned, in the classical approach, partial orders are used to model both specifications and observations of behaviours. On the level of observations (or behaviour instances) posets are used to define operational semantics and are interpreted as 'earlier than' relation with a lack of ordering interpreted as *simultaneity*. On the level of behaviour specifications, posets are usually interpreted as *causality*; a lack of ordering is here interpreted as *independence*. The relationship between these two levels can be described as follows. Let X be a set of event occurrences and let $\prec \subseteq X \times X$ be a (partially ordered) causality relation specifying a given behaviour. Then for every observation o (behaviour instance) we have:

$$a \prec b \;\Rightarrow\; a \prec_o b$$

where $\prec_o$ is the 'earlier than' relation which describes the observation o. Consider the 'not later than' relation on the level of observation. For a given specification (behaviour instance) o, the 'not later than' relation is clearly modelled by $\preceq_o = \prec_o \cup \sim_o$ (since $\sim_o$ is interpreted as simultaneity). We would like *weak causality* to be an abstraction of the 'not later than' relation in the same sense as causality is an abstraction of the 'earlier than' relation. This means that weak causality, denoted by $\sqsubset$, should satisfy the following:

$$a \sqsubset b \;\Rightarrow\; a \preceq_o b$$

for every observation (behaviour instance o). In general, neither $\sqsubset$ can be derived from $\prec$, nor $\prec$ from $\sqsubset$, so we have to consider both of them, i.e. we are looking for axioms defining a relational system $(X, \prec, \sqsubset)$ where $\prec$ is interpreted as causality and $\sqsubset$ as weak causality. We want both $\prec$ and $\preceq$ to be considered as special cases of $\sqsubset$, i.e. we want both $\prec$ and $\preceq$ to satisfy weak causality axioms. But the most important assumption we made is that $\sqsubset$ should be an abstraction of the 'not later than' relations $\preceq_o$, for all observations o of the specified behaviour. From the last assumption it follows that there will be various types of axioms for $\sqsubset$, depending on what kind of observations it abstracts.

3 Directed Extensions of Posets and Order Structures

Let $po = (X, \prec)$ be a poset. An irreflexive relation $\sqsubset$ on X will be called a *directed extension* of $\prec$ if $\prec$ is included in $\sqsubset$ and

$$a \prec b \sqsubset c \;\vee\; a \sqsubset b \prec c \;\Rightarrow\; a \sqsubset c.$$

Note that $\prec$ is the smallest, while $\preceq$ is the greatest directed extension of $\prec$. Moreover, $a \prec b$ implies $b \not\sqsubset a$. We are now ready to introduce the central definition of this paper. By an *order structure* we mean a triple $S = (X, \prec, \sqsubset)$ such that $(X, \prec)$ is a poset and $\sqsubset$ is a directed extension of $\prec$. That is, if the following are satisfied:

OS1 $a \not\sqsubset a$	OS3 $a \prec b \;\Rightarrow\; a \sqsubset b$
OS2 $a \prec b \prec c \;\Rightarrow\; a \prec c$	OS4 $a \prec b \sqsubset c \;\vee\; a \sqsubset b \prec c \;\Rightarrow\; a \sqsubset c$.

We observe that $a \prec b$ implies $b \not\sqsubset a$ (by OS1 and OS4). Note that $a \not\prec a$ follows from OS1 and OS3, and for every poset $(X, \prec)$, $(X, \prec, \prec)$ and $(X, \prec, \preceq)$ are order structures. The latter will subsequently be denoted by $\mathcal{S}(po)$.

Let $S = (X, \prec, \sqsubset)$ be an order structure and $po = (X, \prec_{po})$ be a poset such that $\prec \subseteq \prec_{po}$ and $\sqsubset \subseteq \preceq_{po}$. Then po is called a *poset extension* of S, $po \in posets(S)$. A relational structure $(X, \prec', \sqsubset')$ is an *extension* of S if $\prec \subseteq \prec'$ and $\sqsubset \subseteq \sqsubset'$. Note that for every order structure $(X, \prec, \sqsubset)$, $\mathcal{S}(X, \prec) = (X, \prec, \preceq)$ is its extension. The *intersection* of relational structures $RS_i = (X, R_1^i, R_2^i)$, for $i \in I$, is defined component-wise.

Lemma 2. *Let $S = (X, \prec, \sqsubset)$ be an order structure, and Q be a non-empty set of poset extensions of S. Then $S = \bigcap_{po \in Q} \mathcal{S}(po)$ if and only if the following are satisfied for all distinct $a, b \in X$:*

1. $a \not\prec b \;\Rightarrow\; (\exists po \in Q)\; b \preceq_{po} a$.
2. $a \not\sqsubset b \;\Rightarrow\; (\exists po \in Q)\; b \prec_{po} a$.

Proof. To prove the $(\Rightarrow)$ implication we first observe that if $a \neq b$ then

$$((\forall po \in Q)\; b \not\preceq_{po} a) \;\Rightarrow\; ((\forall po \in Q)\; a \prec_{po} b) \;\Rightarrow\; a \prec b.$$

Thus (1) holds. Similarly, (2) holds since if $a \neq b$ then:

$$((\forall po \in Q)\; b \not\prec_{po} a) \;\Rightarrow\; ((\forall po \in Q)\; a \preceq_{po} b) \;\Rightarrow\; a \sqsubset b.$$

To show the reverse implication, let

$$\bigcap_{po \in Q} \mathcal{S}(po) = (X, \prec', \sqsubset') = (X, \bigcap_{po \in Q} \prec_{po}, \bigcap_{po \in Q} \preceq_{po}).$$

For every $po \in Q$, $\prec \subseteq \prec_{po}$ and $\sqsubset \subseteq \preceq_{po}$. Hence $\prec \subseteq \prec'$ and $\sqsubset \subseteq \sqsubset'$. If $a \prec' b$ and $a \not\prec b$ then, by (1), there is $po \in Q$ such that $b \preceq_{po} a$, a contradiction, since $a \prec' b \;\Rightarrow\; a \prec_q b$, for all $q \in Q$. If $a \sqsubset' b$ and $a \not\sqsubset b$, then, by (2), there is $po \in Q$ such that $b \prec_{po} a$, a contradiction, since $a \sqsubset' b \;\Rightarrow\; a \preceq_q b$, for all $q \in Q$. $\square$

4 Interval, Stratified and Total Order Structures

A relational structure $S = (X, \prec, \sqsubset)$ is called

- *interval order structure* if

IS1 $a \not\sqsubset a$	IS4 $a \prec b \sqsubset c \;\vee\; a \sqsubset b \prec c \;\Rightarrow\; a \sqsubset c$
IS2 $a \prec b \prec c \;\Rightarrow\; a \prec c$	IS5 $a \prec b \sqsubset c \prec d \;\Rightarrow\; a \prec d$
IS3 $a \prec b \;\Rightarrow\; a \sqsubset b$	IS6 $a \sqsubset b \prec c \sqsubset d \;\Rightarrow\; a \sqsubset d \;\vee\; a = d$

- *stratified order structure* if

SS1 $a \not\sqsubset a$	SS3 $a \sqsubset b \sqsubset c \;\Rightarrow\; a \sqsubset c \vee a = c$
SS2 $a \prec b \;\Rightarrow\; a \sqsubset b$	SS4 $a \sqsubset b \prec c \vee a \prec b \sqsubset c \;\Rightarrow\; a \prec c$

- *total order structure* if OS1-OS4 hold and $\prec = \sqsubset$.

Note that SS2 and SS4 imply OS2 and OS4. Clearly, every total order structure is stratified, and every stratified order structure is interval.

The axioms IS1-5 follow from [16] where Lamport introduced a model for specification of system behaviour using Einstein's concept of time-space relationship. The idea was that the relativistic view is relevant whenever signal propagation is not negligibly small compared with the execution time of individual operations. A concurrent system itself was defined as a set of operation executions, each operation execution comprising a set of space-time events. Operation executions were ordered using two relations, $\longrightarrow$ and $\longmapsto$, which were based on the ordering of events in Minkowski space. System executions were represented as structures $LS = (X, \longrightarrow, \longmapsto)$, where X was a set of operation executions, and $\longrightarrow, \longmapsto -id_X$ were relations satisfying IS1-5. Structure defined in [16] are often referred to as 'Lamport structures' [1, 2]. The axioms corresponding to IS1-6 were proposed (in Lamport's framework) in [2].

Stratified order structures had been introduced and used to model concurrent behaviours in [8] (under the name of 'prossets' – preorder specification sets) and in [14] (under the name of 'composets'– combined posets). The latter provides a detailed analysis of the finite case. Order structures have been introduced as a natural generalisation of posets. This and the results that follow can be regarded as an informal justification of the terminology proposed above.

Proposition 3. *Let $po = (X, \prec)$ be a poset. Then:*

1. *po is total if and only if $\mathcal{S}(po)$ is total.*
2. *po is stratified if and only if $\mathcal{S}(po)$ is stratified.*
3. *po is interval if and only if $\mathcal{S}(po)$ is interval.*

Proof. (1) is obvious.
(2,$\Rightarrow$) We need to prove SS3 and SS4 for $\mathcal{S}(po)$.
Let $a \preceq b \preceq c$. Since $\sim \cup\, id_X$ and $\prec$ are transitive, the only non-obvious cases are: $a \prec b \sim c$ and $a \sim b \prec c$. Suppose $a \prec b \sim c$, $a \neq c$ and $a \not\preceq c$. Then $c \prec a$ and, consequently, $c \prec b$, a contradiction. Similarly for $a \sim b \prec c$. Hence $a \preceq c$ or $a = c$. Let $a \prec b \preceq c$. Clearly $a \neq c$. If $b \prec c$ then $a \prec c$. If $b \sim c$ then by reasoning as above, we obtain $a \preceq c$. But if $a \sim c$, then $a \sim c \sim b$, i.e. $a \sim b$ or $a = b$, a contradiction, so $a \prec c$. Similarly for $a \preceq b \prec c$.
(2,$\Leftarrow$) Let $a \sim b \sim c$. Then $a \preceq b \preceq c$, which means that $a \preceq c \vee a = c$; and also $c \preceq b \preceq a$, implying $c \preceq a \vee c = a$. Thus $a \sim c \vee c = a$.
(3,$\Rightarrow$) We need to prove IS5 and IS6 for $\mathcal{S}(po)$ (IS4 is obvious). Let $a \prec b \preceq c \prec d$. Since po is an interval order, we have $a \prec d \vee c \prec b$. But $c \prec b \Rightarrow b \not\preceq c$, yielding $a \prec d$. Let $a \preceq b \prec c \preceq d$ and $a \neq d$. Suppose that $a \not\preceq d$, i.e. $d \prec a$. Then $d \prec a \preceq b \prec c$, implying $d \prec c$, by IS5 proven above, a contradiction.
(3,$\Leftarrow$) Let $a \prec b$ and $c \prec d$. Suppose that $a \not\prec d \wedge c \not\prec b$. Due to the transitivity of $\prec$, this means that $a \sim d$ and $b \sim c$. But $b \sim c \Rightarrow b \preceq c$, so $a \prec b \preceq c \prec d$ implies (by IS5) $a \prec d$, a contradiction. □

Proposition 4. *Let Q be a non-empty set of posets with a common domain, and let $S_Q = \bigcap_{po \in Q} \mathcal{S}(po)$. Then:*

1. *S_Q is an order structure.*
2. *If every element of Q is an interval order, then S_Q is interval.*
3. *If every element of Q is a stratified order, then S_Q is stratified.*
4. *If every element of Q is a total order, then S_Q is total.*

Proof. From Proposition 3 it follows that in order to show (1)-(4) it suffices to prove that if order structures R_i, for $i \in I$, satisfy an axiom Ax used in the definition of the various classes of order structures, then so does $R = \bigcap_{i\in I} R_i$. We first observe that each such axiom Ax can be rewritten as (a conjunction of) axiom(s) of the form $\alpha_1 \wedge \ldots \wedge \alpha_k \Rightarrow \beta$, where each α_i is of the form $a \neq b$, $a \prec b$ or $a \sqsubset b$, and β is of the form $false$, $a \prec b$ or $a \sqsubset b$. Indeed, SS1 can be rewritten as $a \sqsubset a \;\Rightarrow\; false$, IS6 as $a \sqsubset b \;\wedge\; b \prec c \;\wedge\; c \sqsubset d \;\wedge\; a \neq d \;\Rightarrow\; a \sqsubset d$, etc. Moreover, the additional axiom for total order structures can be formulated as $a \sqsubset b \;\Rightarrow\; a \prec b$. The proposition now follows easily, since if $\alpha_1 \wedge \ldots \wedge \alpha_k \;\Rightarrow\; \beta$ holds in each R_i, for $i \in I$, then it also holds for $R = \bigcap_{i\in I} R_i$. □

5 Generalisations of Szpilrajn's Theorem

This section contains our main result. For every order structure S, let $intervals(S)$, $strats(S)$ and $totals(S)$ respectively denote the set of all interval, stratified and total order extensions of S. E.g., $totals(S) = \{po \in posets(S) \mid po \text{ is total}\}$.

Theorem 5. *Let $S = (X, \prec, \sqsubset)$ be an order structure.*

1. $S = \bigcap_{po\in posets(S)} \mathcal{S}(po)$.
2. $S = \bigcap_{po\in intervals(S)} \mathcal{S}(po)$ *when S is interval.*
3. $S = \bigcap_{po\in strats(S)} \mathcal{S}(po)$ *when S is stratified.*
4. $S = \bigcap_{po\in totals(S)} \mathcal{S}(po)$ *when S is total.*

In the proof of Theorem 5 we will use Lemma 2 with Q being equal to $posets(S)$, $intervals(S)$ and $strats(S)$. We also need the following result.

Lemma 6 [6, 20]. *Let $po = (X, \prec)$ be a poset and $a, b \in X$, $a \sim b$.*
Define $Y = \{b\} \cup \{y \mid y \prec b\}$ and $Z = \{a\} \cup \{z \mid a \prec z\}$.
Then $(X, \prec^{ba}) = (X, \prec \cup\, Y \times Z)$ is a poset.

Proof of Theorem 5(1,4). We note that (4) is equivalent to Szpilrajn's theorem. To show (1) we first observe that $posets(S) \neq \emptyset$ since $(X, \prec) \in posets(S)$. The poset $(X, \prec)$ also satisfies (1) of Lemma 2, so we only have to prove Lemma 2(2). Let $a \neq b$ and $a \not\sqsubset b$. We may assume $b \not\prec a$, otherwise $(X, \prec)$ has the required property. Hence $a \sim b$. Take $\prec^{ba}$ from Lemma 6. Clearly $\prec \;\subseteq\; \prec^{ba}$. We will show that $\sqsubset \;\subseteq\; \preceq^{ba}$. Suppose that $c \sqsubset d \prec^{ba} c$. This means that $c \sqsubset d$ and $(d, c) \in Y \times Z$. Since $a \not\sqsubset b$, we have $a \neq c$ or $b \neq d$. Suppose $a \neq c$. By the definition of Z and OS4 we have $a \sqsubset d$. Hence, by $d \in Y$ and OS4, $a \sqsubset b$, a contradiction. Thus, $c \sqsubset d \;\Rightarrow\; d \not\prec^{ba} c \;\Rightarrow\; c \preceq^{ba} d$. □

6 Representation Theorems and Their Applications

The idea of *representation* lies in the use of simpler objects to represent more complex ones. It turns out that both interval orders and stratified orders can be represented by total orders which clearly are simpler structures. In the same way both interval order structures and stratified order structures can be represented by partial orders.

Proposition 7. *A poset $po = (X, \prec)$ is stratified if and only if there is a total order $(T, \lhd)$ and a mapping $\phi : X \to T$ such that for all $a, b \in X$: $a \prec b \Leftrightarrow \phi(a) \lhd \phi(b)$.*

Theorem 8 [5]. *A poset $po = (X, \prec)$ is interval if and only if there is a total order $(T, \lhd)$ and two mappings $\phi, \psi : X \to T$ such that for all $a, b \in X$, $\phi(a) \lhd \psi(a)$ and $a \prec b \Leftrightarrow \psi(a) \lhd \phi(b)$.*

An elegant proof of Theorem 8 can be found in [6]. Its weaker version had been known to Russell and Wiener (see [1]). Its discrete version (when $(T, \lhd)$ is the arithmetic order of integer numbers) was formulated and proven in [12].

We now will obtain representation theorems for order structures, and next use them to prove the remaining parts of Theorem 5.

Theorem 9 [1]. *Let $S = (X, \prec, \sqsubset)$ be an interval order structure. Then there is a partial order $(T, \lhd)$ and two mappings $\phi, \psi : X \to T$ such that for all distinct $a, b \in X$, $\phi(a) \lhd \psi(a)$ and the following hold:*

$$\begin{aligned} a \prec b &\Leftrightarrow \psi(a) \lhd \phi(b) \\ a \sqsubset b &\Leftrightarrow \phi(a) \lhd \psi(b) \vee \phi(a) = \psi(b). \end{aligned}$$

Proof of Theorem 5(2). First we have to show that $intervals(S) \neq \emptyset$. Let $(T, \lhd_t)$ be any total extension of the poset $(T, \lhd)$ from Theorem 9. Define $po = (X, \prec_{po})$ where $a \prec_{po} b \Leftrightarrow \psi(a) \lhd_t \phi(b)$. By Theorem 8, po is an interval order. Moreover, $po \in intervals(S)$, since for all distinct $a, b \in X$:

$$\begin{aligned} a \prec b &\Rightarrow \psi(a) \lhd \phi(b) \Rightarrow \psi(a) \lhd_t \phi(b) \Rightarrow a \prec_{po} b \\ a \sqsubset b &\Rightarrow \phi(a) \lhd \psi(b) \vee \phi(a) = \psi(b) \\ &\Rightarrow \phi(a) \lhd_t \phi(b) \vee \phi(a) = \psi(b) \\ &\Rightarrow \psi(b) \not\lhd_t \phi(a) \Rightarrow b \not\prec_{po} a \Rightarrow a \preceq_{po} b \end{aligned}$$

We will now prove that the conditions (1) and (2) of Lemma 2 are satisfied. Take $(T, \lhd)$ from Theorem 9. Suppose $x, y \in T$ are such that $x \sim_{\lhd} y$. Take then $(T, \lhd^{xy})$, where $\lhd^{xy}$ is constructed from $\lhd$ as in Lemma 6. Let $(T, \lhd_t^{xy})$ be any total extension of $(T, \lhd^{xy})$. Clearly $x \lhd_t^{xy} y$. Define $po_{xy} = (X, \prec_{xy})$ by:

$$c \prec_{xy} d \Rightarrow \psi(c) \lhd_t^{xy} \phi(d).$$

By Theorem 8, po_{xy} is an interval order, and proceeding similarly as above, one may show that $po_{xy} \in intervals(S)$.
Suppose $a \not\prec b$ and $a \neq b$. We may assume $b \not\sqsubset a$, otherwise every element of $intervals(S)$ satisfies the required property. But this means that $a \sim b$. Let $x = \phi(b)$, $y = \psi(a)$. Then $x \sim_{\lhd} y$, so we can define po_{xy} as above. We have $po_{xy} \in intervals(S)$. Moreover:

$$x \lhd_t^{xy} y \Rightarrow \phi(b) \lhd_t^{xy} \psi(a) \Rightarrow \psi(a) \not\lhd_t^{xy} \phi(b) \Rightarrow a \not\prec_{xy} b \Rightarrow b \preceq_{xy} a.$$

Suppose $a \not\sqsubset b$. Similarly as above we may assume $a \sim b$. Let $x = \psi(b)$, $y = \phi(a)$. Hence $x \sim_{\lhd} y$ and $po_{xy} \in intervals(S)$. In this case we have:

$$x \lhd_t^{xy} y \Rightarrow \psi(b) \lhd_t^{xy} \phi(a) \Rightarrow b \prec_{xy} a.$$

□

The fact that the conditions (1) and (2) of Lemma 2 hold when S is an interval order structure and $Q = \mathit{intervals}(S)$ had in fact been proven in [1], using different technique and motivation. Our proof is simpler and in more general framework. [1] deals exclusively with Lamport structures (sometimes augmented with axiom IS6). The equivalence of Theorem 5(2) is also indirectly formulated in [1], however no proof is given.

Theorem 10 [15]. *Let $S = (X, \prec, \sqsubset)$ be a stratified order structure. Then there is a partial order $(T, \lhd)$ and a mapping $\phi : X \to T$ such that for all distinct $a, b \in X$:*

$$\begin{array}{l} a \prec b \;\Rightarrow\; \phi(a) \lhd \phi(b) \\ a \sqsubset b \;\Leftrightarrow\; \phi(a) \lhd \phi(b) \vee \phi(a) = \phi(b). \end{array}$$

Because in Theorem 10 we have '$a \prec b \;\Rightarrow$' instead of a stronger '$a \prec b \;\Leftrightarrow$', it has a limited applicability. To prove Theorem 5(3) we need another result.

Lemma 11. *Let $S = (X, \prec, \sqsubset)$ be a stratified order structure, $a, b \in X$, $a \neq b$,*

$$\begin{array}{ll} Y = \{b\} \cup \{c \mid c \sqsubset b\} & W = \{c \mid c \prec b\} \\ Z = \{a\} \cup \{c \mid a \sqsubset c\} & V = \{c \mid a \prec c\}. \\ \prec_1 = \prec \cup\, W \times Z \cup Y \times V & \sqsubset_1 = \sqsubset \cup\, Y \times Z - id_{Y \cap Z} \\ \prec_2 = \prec \cup\, Y \times Z & \sqsubset_2 = \sqsubset \cup\, Y \times Z \end{array}$$

1. *If $a \not\prec b$ and $b \not\sqsubset a$ then $S_1 = (X, \prec_1, \sqsubset_1)$ is a stratified order structure.*
2. *If $a \not\sqsubset b$ and $b \not\prec a$ then $S_2 = (X, \prec_2, \sqsubset_2)$ is a stratified order structure.*

Proof. (1) Let $\prec_0 = \prec_1 - \prec$ and $\sqsubset_0 = \sqsubset_1 - \sqsubset$. Clearly, $\prec_1$ and $\sqsubset_1$ are irreflexive (by $a \not\prec b$, $V \cap Y = W \cap Z = \emptyset$) and $\prec_1 \subseteq \sqsubset_1$.
To show $c \sqsubset_1 d \sqsubset_1 e \;\Rightarrow\; c \sqsubset_1 e \;\vee\; c = e$ we have:

$$\begin{array}{ll} c \sqsubset_0 d \sqsubset_0 e & \Rightarrow\; (c, d, e) \in Y \times (Z \cap Y) \times Z \\ c \sqsubset d \sqsubset_0 e \vee c \sqsubset_0 d \sqsubset e & \Rightarrow\; (c, d, e) \in Y \times Y \times Z \cup Y \times Z \times Z. \end{array}$$

To show $c \sqsubset_1 d \prec_1 e \;\Rightarrow\; c \prec_1 e$ we have:

$$\begin{array}{ll} c \sqsubset_0 d \prec_0 e & \Rightarrow\; (c, d, e) \in Y \times (Z \cap Y) \times V \\ c \sqsubset d \prec_0 e & \Rightarrow\; (c, d, e) \in W \times W \times Z \cup Y \times Y \times V \\ c \sqsubset_0 d \prec e & \Rightarrow\; (c, d, e) \in Y \times Z \times V. \end{array}$$

To show $c \prec_1 d \sqsubset_1 e \;\Rightarrow\; c \prec_1 e$ we have:

$$\begin{array}{ll} c \prec_0 d \sqsubset_0 e & \Rightarrow\; (c, d, e) \in W \times (Z \cap Y) \times Z \\ c \prec d \sqsubset_0 e & \Rightarrow\; (c, d, e) \in W \times Y \times Z \\ c \prec_0 d \sqsubset e & \Rightarrow\; (c, d, e) \in W \times Z \times Z \cup Y \times V \times V. \end{array}$$

(2) We observe that $Y \cap Z = \emptyset$ since $a \not\sqsubset b$. Thus $\prec_2$ and $\sqsubset_2$ are irreflexive. If $c \sqsubset_2 d \sqsubset_2 e$ then, by $Y \cap Z = \emptyset$, we must have $c \sqsubset d$ or $d \sqsubset e$. Moreover, if $c \not\sqsubset d$ or $d \not\sqsubset e$ then $c \prec e$ holds. Hence SS2-4 hold. □

Proof of Theorem 5(3).[3] First we have to show that $strats(S) \neq \emptyset$. Let $(T, \lhd_t)$ be any total extension of $(T, \lhd)$ from Theorem 10. Define $po = (X, \prec_{po})$, where $a \prec_{po} b \Leftrightarrow \phi(a) \lhd_t \phi(b)$. By Proposition 7, po is a stratified poset. Moreover, for all distinct $a, b \in X$:

$$\begin{aligned}
a \prec b &\Rightarrow \phi(a) \lhd \phi(b) \Rightarrow \phi(a) \lhd_t \phi(b) \Rightarrow a \prec_{po} b \\
a \sqsubset b &\Rightarrow \phi(a) \lhd \phi(b) \vee \phi(a) = \phi(b) \\
&\Rightarrow \phi(a) \lhd_t \phi(b) \vee \phi(a) = \phi(b) \\
&\Rightarrow \phi(b) \not\lhd_t \phi(a) \Rightarrow b \not\prec_{po} a \Rightarrow a \preceq_{po} b.
\end{aligned}$$

Hence $po \in strats(S)$. We will now prove that the conditions (1) and (2) of Lemma 2 are satisfied.
Suppose $a \not\prec b$. We assume $b \not\sqsubset a$, otherwise every element of $strats(S)$ satisfies the property. Take S_1 from Lemma 11(1). Since S_1 is an extension of S, $strats(S_1) \subseteq strats(S)$. Note that $b \sqsubset_1 a$, i.e. $b \preceq_{po} a$, for every $po \in strats(S_1) \neq \emptyset$.
Suppose $a \not\sqsubset b$. We now can assume that $b \not\prec a$. Take S_2 from Lemma 11(2). Clearly, we have $strats(S_2) \subseteq strats(S)$. Note that $b \prec_2 a$, i.e. $b \prec_{po} a$, for every $po \in strats(S_2) \neq \emptyset$. □

7 Concluding Remarks

The main result, Theorem 5, can be interpreted as follows: (1) If all observations are modelled by total orders, posets alone (posets are equivalent to total order structures !) are sufficient to specify concurrent behaviours. (2) If all observations are modelled by stratified orders, stratified order structures are sufficient to specify concurrent behaviours and should be used instead of posets. (3) If all observations are modelled by interval orders (and this is becoming an increasingly popular approach), interval order structures could be used to specify concurrent behaviours. (4) If observations are modelled by general poset, one should consider general order structures.

These could be seen as an additional support for the approach presented in [13], where causality is treated as only one of four fundamental invariants characterising concurrent behaviours. For an order structure $(X, \prec, \sqsubset)$ we propose to interpret $\prec$ as causality and $\sqsubset$ as 'weak' causality.

[1, 8, 14, 16] show how relational structures analysed in this paper could be used to specify concurrent behaviours of distributed protocols, inhibitor Petri nets, etc. This paper provides a general framework for and builds a bridge between the approaches of [1, 2, 8, 13, 14, 16].

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[3] For the finite case, Theorem 5(3) was proven in [14] using a different technique.

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